Shape dependence of Mutual information in the OPE limit.

arXiv: 2207.05268 w/ Liangyu Chen.

Introduction / Motivation

- Entanglement structure in QFTs

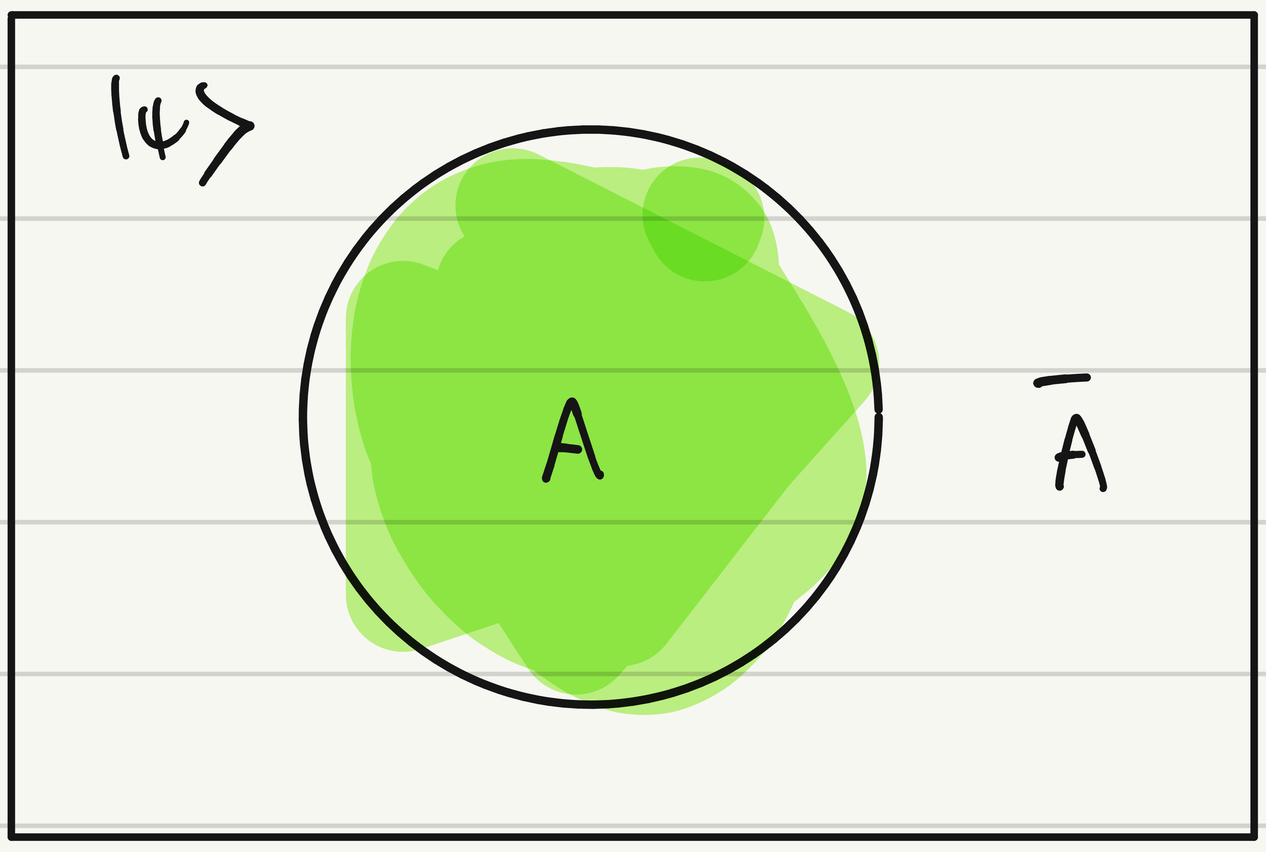
  (i) order parameter for topological phases

  (ii) C - theorem for RG flows

  (iii) geometries in AdS/CFT : RT formula

— probes of entanglement structure :

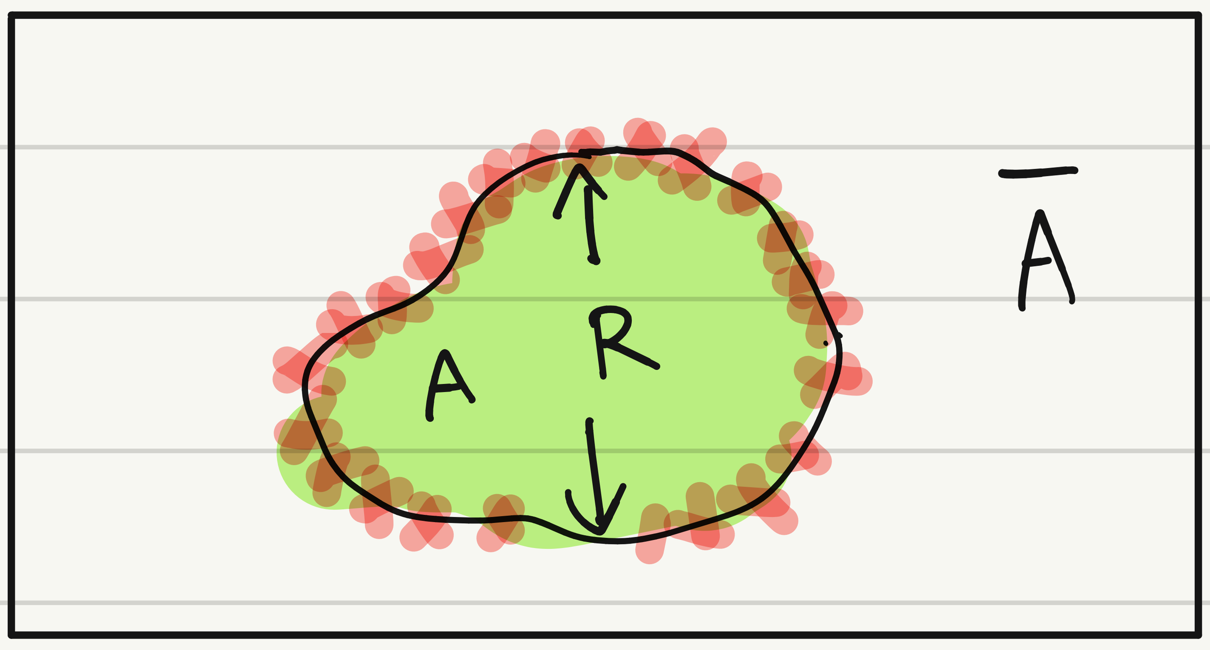
* Entanglement entropy (EE)



$$\rho_A^\psi = \mathrm{tr}_{\bar{A}} |\psi\rangle\langle\psi|$$

$$S_A^\psi = -\mathrm{tr}\, \rho_A^\psi \cdot \ln \rho_A^\psi$$

* EE : depends on UV-cut-off (δ)



source of UV-divergence :

short-range entanglement cross ∂A

e.g. $|\Omega\rangle$ in CFT

$$S_A = C_{d-2} \frac{R^{d-2}}{\delta^{d-2}} + C_{d-4} \frac{R^{d-4}}{\delta^{d-4}} + \cdots + \begin{cases} C_{-1} & d = \text{odd} \\ C_0 \ln\left(\frac{R}{\delta}\right) & d = \text{even} \end{cases}$$

— $\{C_{d-2}, C_{d-4}, \ldots\}$ UV-dependent

— $\{C_{-1}, C_0\}$ encodes universal info.

— by extracting universal data, can learn more about novel aspects of QFTs

* UV-independent entanglement measures

— topological EE (order parameter of top. phase)

* $S_{top.} = S_A + S_B + S_C - S_{AB} - S_{BC} - S_{AC} + S_{ABC}$

— relative entropy $S_A^\psi(\rho|\sigma)$

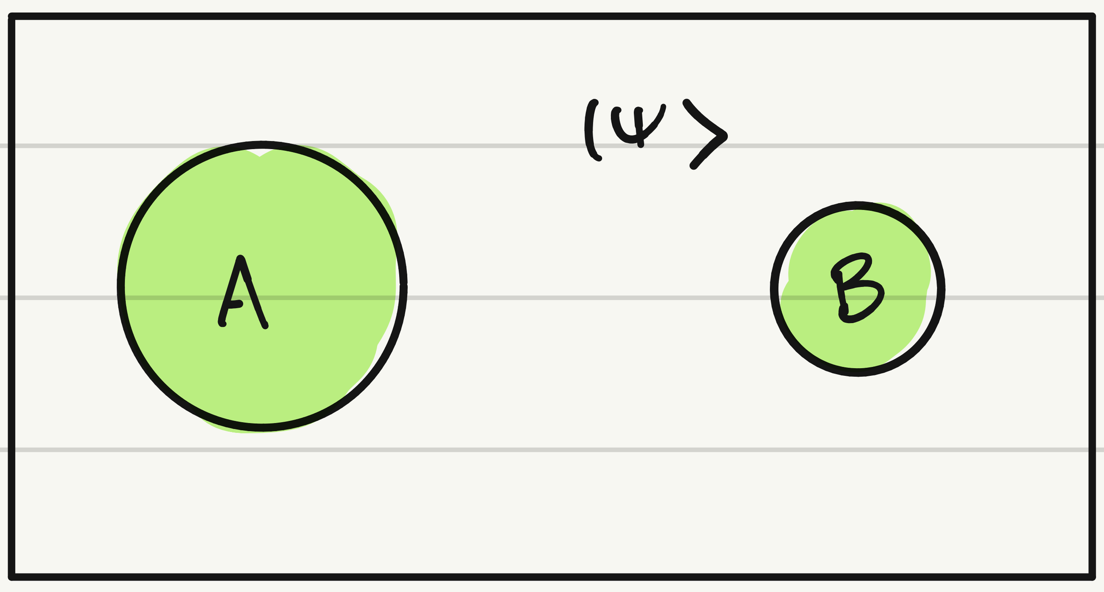
* $S_A^\psi(\rho|\sigma) = \mathrm{tr}\, \rho_A^\psi \ln(\rho_A^\psi / \rho_A^\sigma)$

* measure of distinguishability between $\rho_A^\psi / \rho_A^\sigma$

* monotonicity of $S_A^\psi(\rho|\sigma)$ under $A \to \tilde{A} \subseteq A$

proof. of ANEC, QNEC, etc.

— Mutual information (MI)



$I_{A,B}^\psi = S_{A\cup B}^\psi - S_A^\psi - S_B^\psi$

measures correlation (classical & quantum) between A and B in $|\psi\rangle$

* Measure of entanglement in mixed-states

* $I_{A,B}^\psi$ is monotonous : $I_{A,B}^\psi \leq I_{\hat{A},B}$

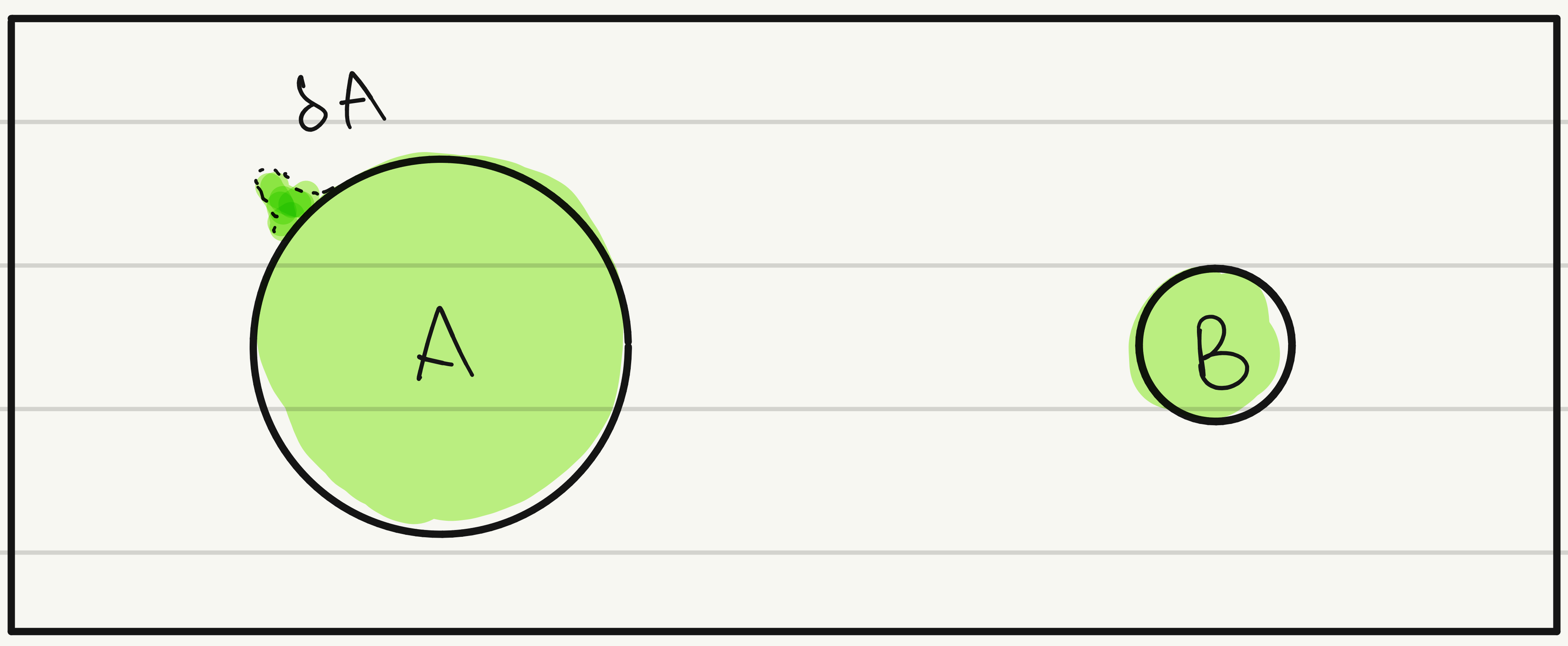
$A \subseteq \hat{A}$

— Shape dependence of entanglement measures.

$S(A)$
- cusp contribution (universal)
- perturbation theory about spheres.
  - — minimization
  - — universal entanglement density
- as control parameter: RG-flow

— In this talk: shape dependence of MI

* linear response about spheres



$$\delta I_{A,B} = I_{A+\delta A,B} - I_{A,B}.$$

* unlike EE on spheres, MI between spheres are not known exactly

* we will work in OPE limit:

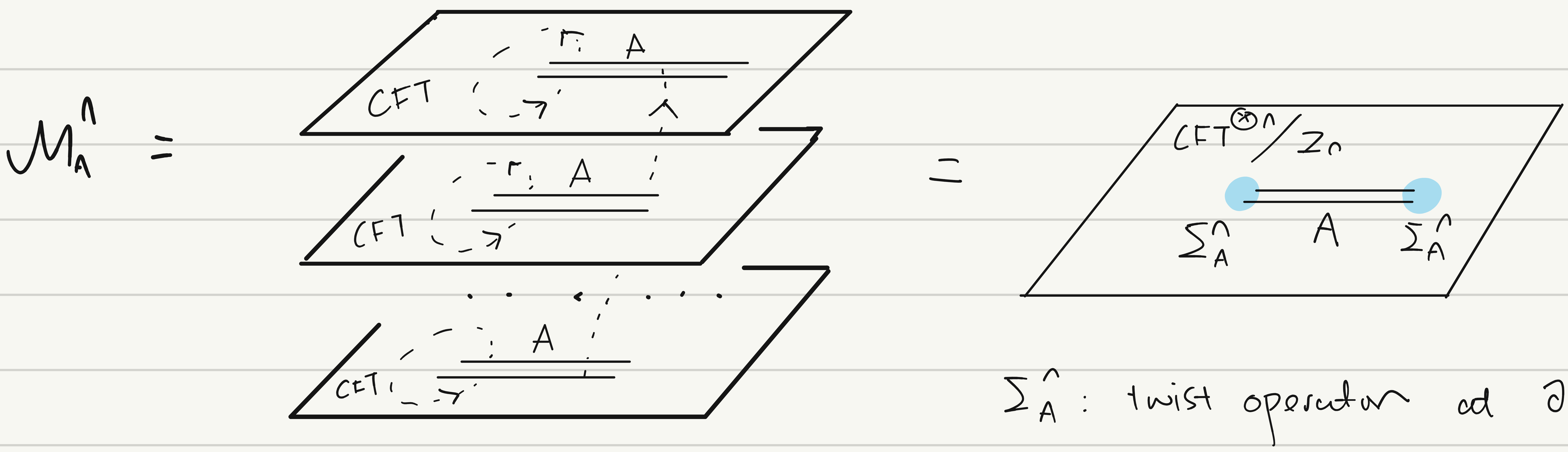
$$r = \frac{R_A R_B}{(L - R_A - R_B)^2} \ll 1$$

Main Tools :

① Replica trick :

$$S_A = -\mathrm{tr}_A \rho_A . \ln \rho_A = \lim_{n\to 1}\left(\frac{1}{1-n}\,\mathrm{tr}_{\hat A}\, \rho_A^n\right).$$

$$\mathrm{tr}_A(\rho_A^n) = Z(\mathcal{M}_A^n) = \langle \Sigma_A^n \rangle_{/\mathrm{CFT}^{\otimes n}/Z_n}$$



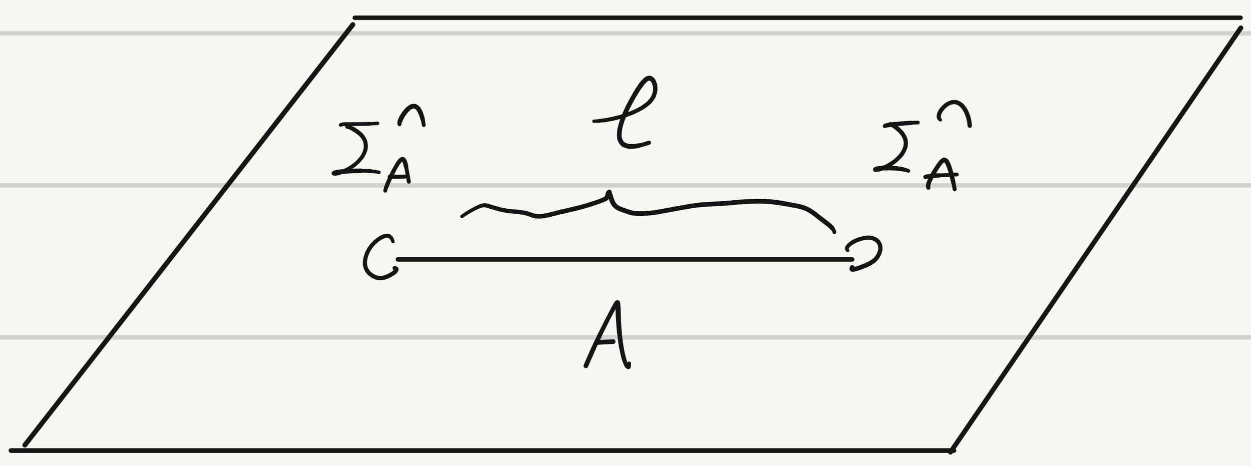
Σ_A^n : twist operator at ∂A

— in 2D CFTs, Σ_A^n are "quasi-local" primary operator

$$\Delta_n = \frac{c}{6}\left(1-\frac{1}{n^2}\right)$$

$$S_A = \lim_{n\to 1}\frac{1}{1-n}\ln\langle \Sigma_A^n\, \Sigma_A^n\rangle$$

$$= \frac{c}{3}\ln\left(\ell/\delta\right)$$

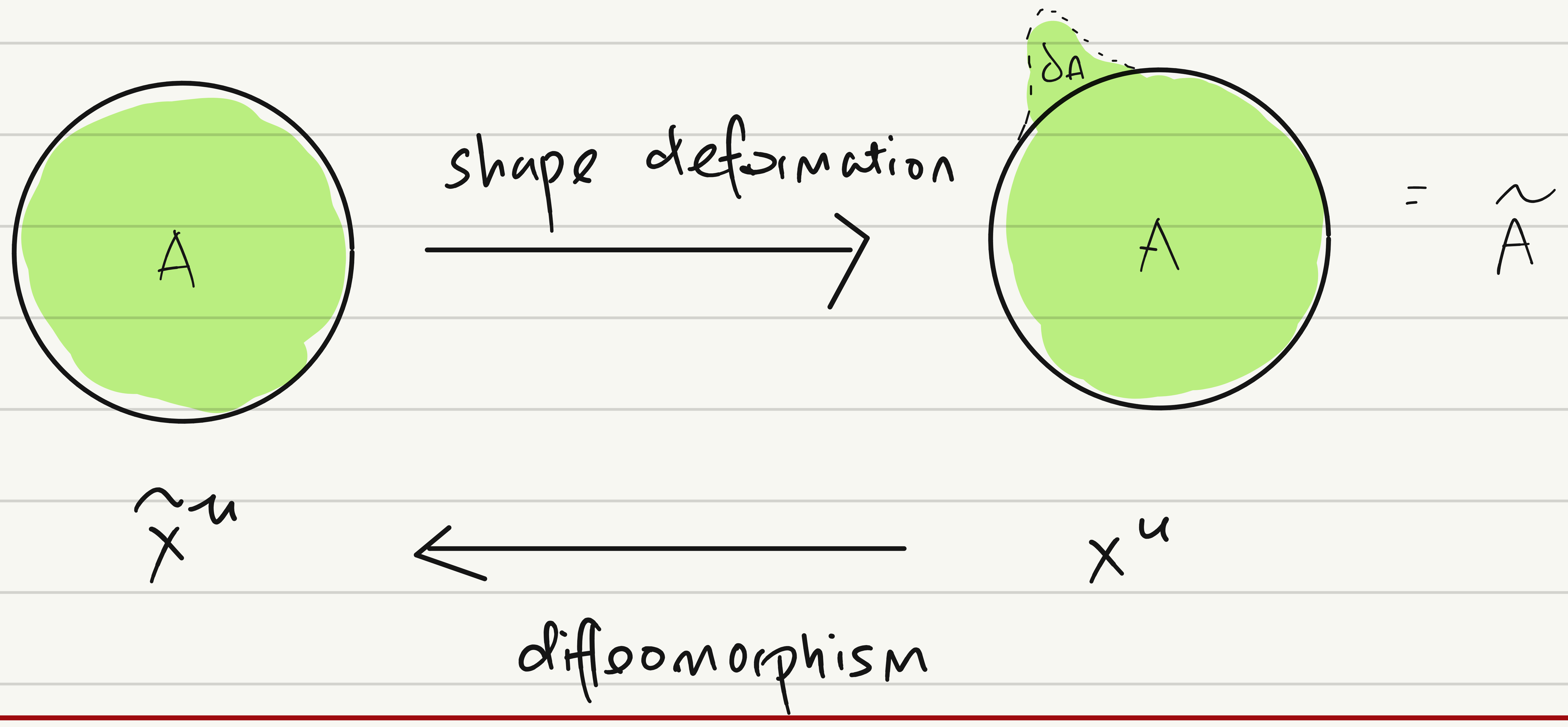


② Shape perturbation theory :

$\delta I_{A,B} \longrightarrow \delta S_A \longrightarrow \delta\rho_A$ to 1st order in

$A \to \hat{A} = A + \delta A$?

Shape perturbation theory (cont'ed)



$$[\rho_{\tilde{A}}(g)]^{\alpha}_{\beta} = \int [D\phi]^{\alpha}_{\beta}\, e^{-S_E(g,\hat{A})} = \int [D\phi]^{\tilde{\alpha}}_{\tilde{\beta}}\, e^{-S_E(\tilde{g},A)}$$

$$= [\rho_A(\tilde{g})]^{\tilde{\alpha}}_{\tilde{\beta}}$$

$$\tilde{g}_{\mu\nu} = g_{\alpha\beta}\frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu}$$

$$= g_{\mu\nu} + \nabla_\mu \xi_\nu$$

where $\tilde{x}^\mu = x^\mu + \xi^\mu$

$$\therefore \quad \delta\rho_A(g) = \rho_{\hat{A}}(g) - \rho_A(g) = \rho_A(\tilde{g}) - \rho_A(g)$$

$$= \delta g[\rho_A(g)] = \int \delta g_{\mu\nu} T^{\mu\nu}(x) dx = \int \nabla_\nu \xi_\nu . T^{\mu\nu}(x) dx$$

---

③ Entanglement 1st law

$\rho_A \to \rho_A + \delta\rho_A$: $S_A \to S_A + ?$

— Modular Hamiltonian : $\hat{\rho}_A = e^{-2\pi . \hat{K}_A}$

— Relative entropy : $S_A(\rho_A | \rho_A + \delta\rho_A)$

Entanglement 1st law (cont'ed)

— can be written :

$$S(\rho|\rho+\delta\rho) = \mathrm{tr}\, \delta\rho \cdot K + S(\rho) - S(\rho+\delta\rho)$$

— relative entropy measures "distinguishability"

$\therefore$ $S(\rho|\rho+\delta\rho) \geq 0$ $\forall\, \delta\rho$, linear response = 0.

$$\therefore \quad \boxed{S(\rho+\delta\rho) - S(\rho) = \mathrm{tr}.\ \delta\rho.\ K}$$

entanglement 1st law

+ RT formula $\longrightarrow$ Einstein eqn. in AdS/CFT, (Faulkner, 15)...

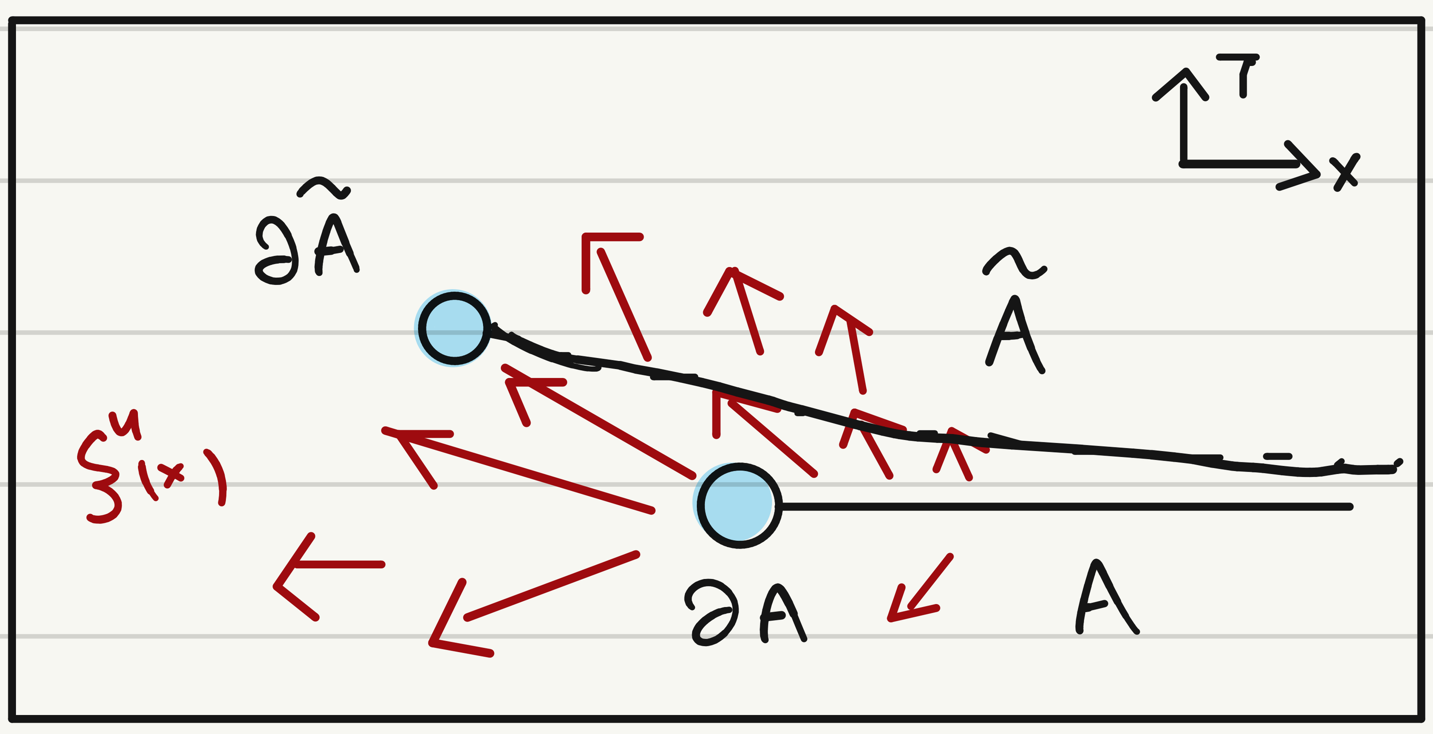
Shape perturbation theory + entanglement 1st law:

$$\delta S_A = S(\rho_{\hat{A}}) - S(\rho_A) = \mathrm{tr}(\delta\rho_A)\, K$$

$$= \int \nabla_u \xi_v(x) . dv . \mathrm{tr}\left(T^{uv}(x). K\right).$$

Recall : $\xi_u(x)$ deformation vector field in (space-time).

s.t. $\quad \partial A + \xi \longrightarrow \partial\hat{A}$

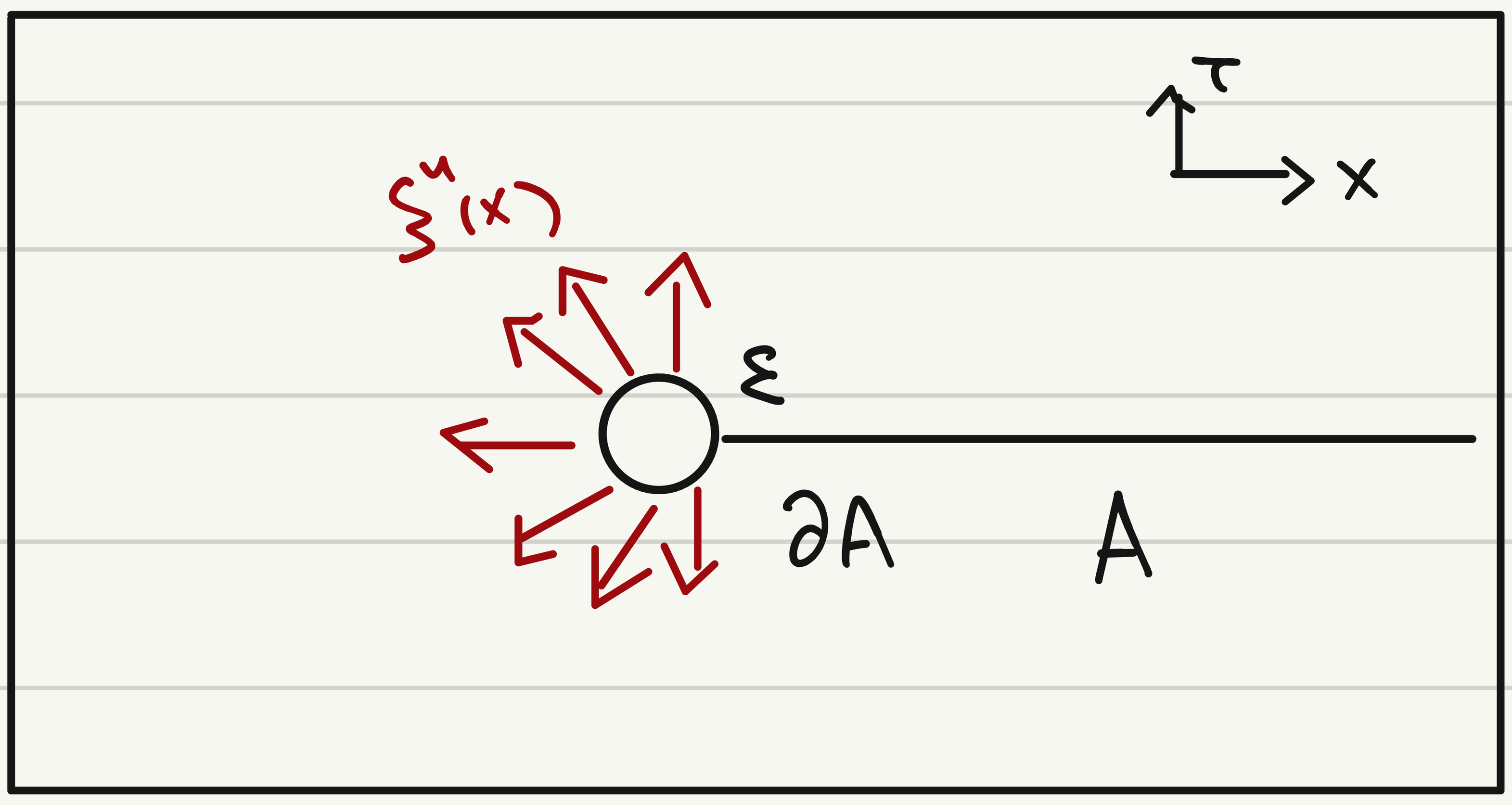


Shape perturbation theory + entanglement 1st law (cont'd).

$$\delta S_A = \int \nabla_\mu \xi_\nu \langle T^{\mu\nu}(x)\, K \rangle.\, dx + O(\xi^2)$$

$$= \underline{\oint_\varepsilon n^\mu \xi^\nu \langle T_{\mu\nu}(x)\, K \rangle\, dx} - \int \xi_\nu . \langle \nabla_\mu T^{\mu\nu}(x)\, K \rangle dx$$

$\qquad\qquad = 0$ ward-identity



* only $\xi^\mu(x)$ at ∂A matters
* $\oint_\varepsilon$ required by cut-off
* can assume $\xi^\mu(x) =$ const on $\oint_\varepsilon$

— can write $\oint_\varepsilon$ in $(z, \bar{z})$ coordinates

$$\delta S_A = \xi^z . \oint dz \langle T_{zz}(z, \bar{z}, y).\, K \rangle + \xi^{\bar{z}} \oint d\bar{z} \langle T_{z\bar{z}}(..)\, K \rangle$$
$$+ \{ z \to \bar{z} \}.$$

$\therefore$ Key point : δS_A is captured by poles in

$$\langle T_{zz}(z, \bar{z}, y)\, K \rangle = \dots + \frac{\#?}{z - \text{“}\partial A\text{”}} + \dots$$

$$\langle T_{\bar{z}\bar{z}}(z, \bar{z}, y)\, K \rangle = \dots + \frac{\#?}{\bar{z} - \text{“}\partial A\text{”}} + \dots$$

We only need to extract the relevant residues #'s.

h.c

Warming up: Shape-dependence of 1-interval EE.

τ, x $\qquad$ $|\Omega\rangle$

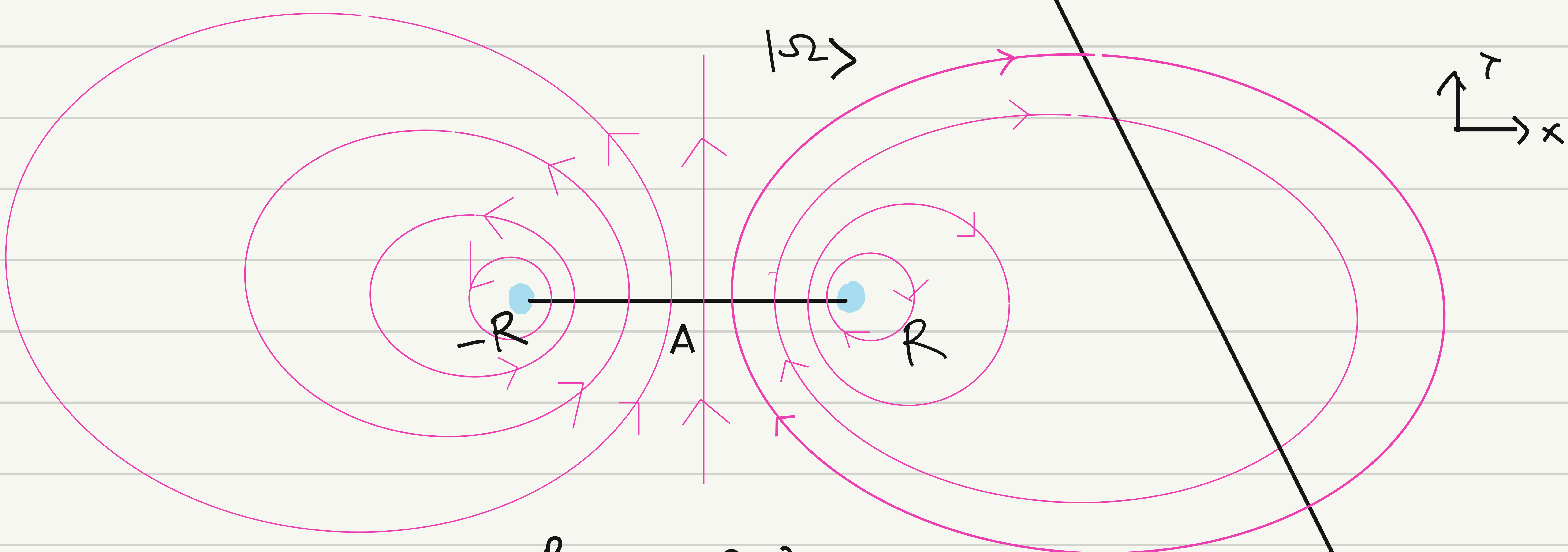
δA : moving ● by ←

shape perturbation theory + entanglement 1st law

$$\delta S_A = \xi^z \oint dz . \langle T_{zz} K_A^\Omega \rangle + h.c. \; ; \qquad \hat{\rho}_A^\Omega = e^{-2\pi \hat{K}_A^\Omega}$$

$$K_A^\Omega = \int_A dx \left[\frac{R^2 - x^2}{2R} T_{00}(x) \right] \qquad T_{00} = T_{zz} + T_{\bar z \bar z}$$

K_A^Ω is explicitly known : (conformal) isometry of $|\Omega\rangle$.



$$\delta S_A = \xi^z \oint dz . \int_{-R}^{R} dx \left(\frac{R^2 - x^2}{2R} \right) \underbrace{\langle T_{zz}(z) T_{zz}(x) \rangle}_{= \frac{c/2}{(z-x)^4} + \cdots} + h.c.$$

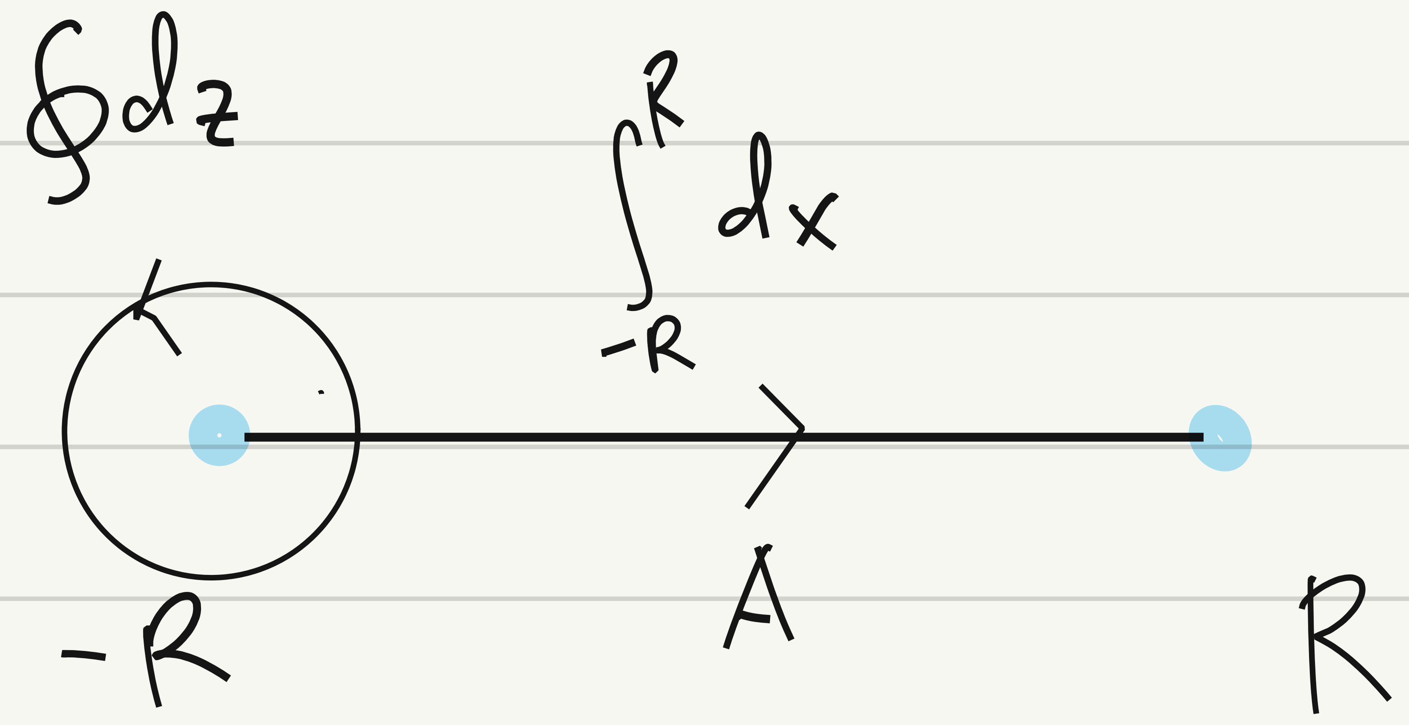
$$= \xi^z \oint dz \left\{ \frac{c}{6(z+R)^2} + \frac{c}{6R(z+R)} + \cdots \right\} = \frac{c}{6R} \xi^z \quad (\checkmark)$$

$z - "\partial A" = z + R$; $\qquad S_A = \frac{c}{3} \ln\left(\frac{2R}{\delta}\right)$; $2R \to 2R + \xi^z$

Warming up: Shape-dependence of 1-interval EE. (cont'd)

comment:

— integration order is important!



$\oint dz \int dx$ (✓) v.s. $\int dx \oint dz$ (✗)

$\nexists$ simple pole $\frac{1}{z+R}$

$\Rightarrow$ gives $\emptyset$!

— Simple pole $\propto \frac{1}{z+R}$ only emerges after $\int dx$ integral, not directly from the integrand...

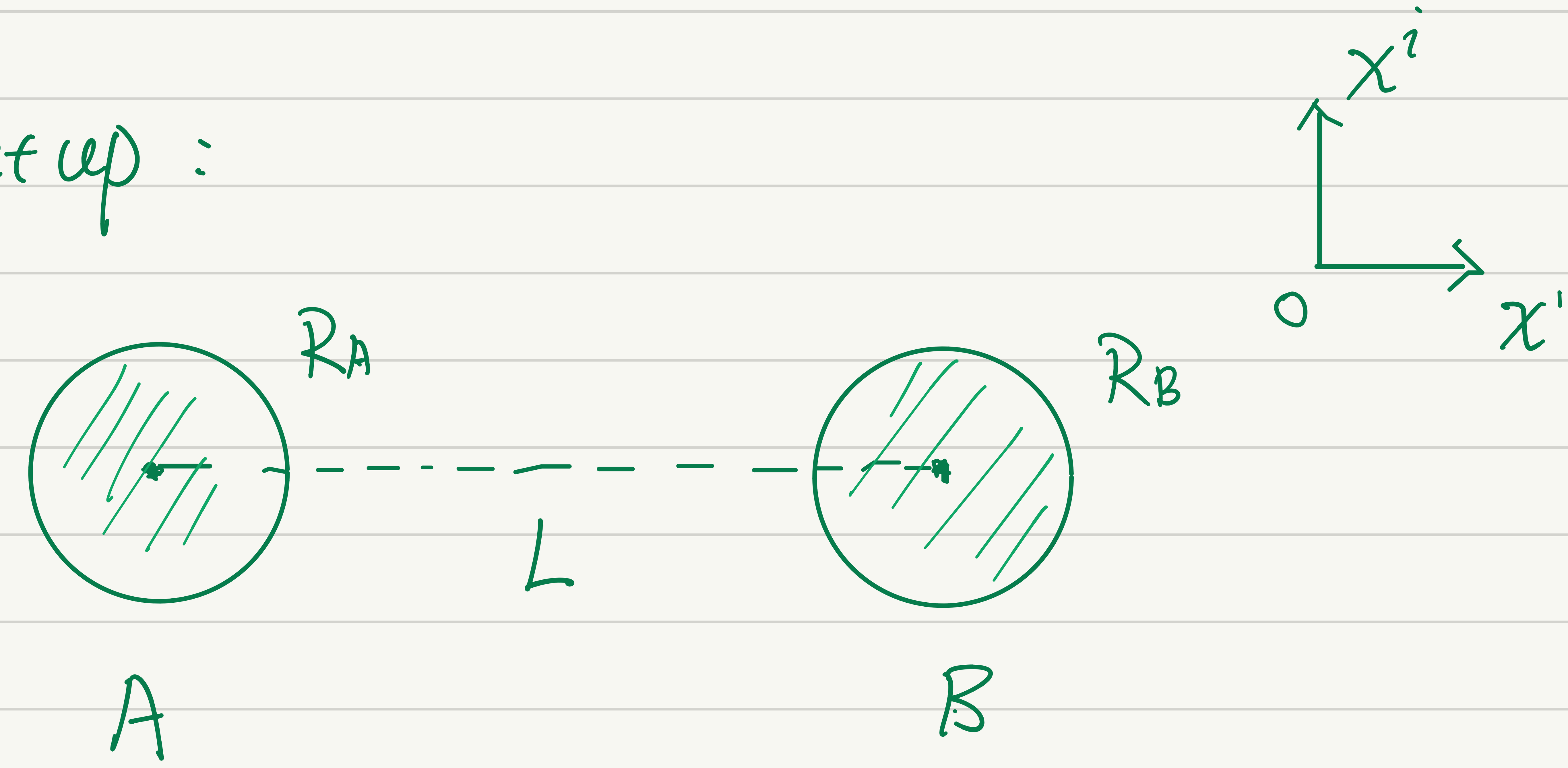
— Shape response is produced "collectively" by non-local correlation of $K_A = \int_{-R}^{R} dx (\ldots) T_{00}(x)$.

— important lesson for $\delta I_{A,B}$!

The shape dependence of mutual information.

1. Mutual info and modular Hamiltonian in the OPE limit.

Setup :



1.1 The replica trick.

Rényi mutual info :

$$\boxed{\begin{aligned} I_n(A,B) &= S_n(A) + S_n(B) - S_n(A\cup B) \\ &= \frac{1}{n-1} \ln \frac{\langle \Sigma_n^A \Sigma_n^B \rangle_{\mathcal{M}^n}}{\langle \Sigma_n^A \rangle_{\mathcal{M}^n} \langle \Sigma_n^B \rangle_{\mathcal{M}^n}} \end{aligned}}$$

where Σ_n^A, Σ_n^B are twist operators.

Assuming R_B is small enough, thus we can take the OPE of twist operator Σ_n^B, i.e.

$$\boxed{\Sigma_n^B = \langle \Sigma_n^B \rangle_{\mathcal{M}^n} \left(1 + \frac{1}{2}(2R_B)^{2\Delta} \times \sum_{j \neq k}^{n-1} C_{j-k}\, \mathcal{O}^{(j)} \mathcal{O}^{(k)} + \cdots \right)}$$

where Δ is the scaling dimension of operator $\mathcal{O}$.

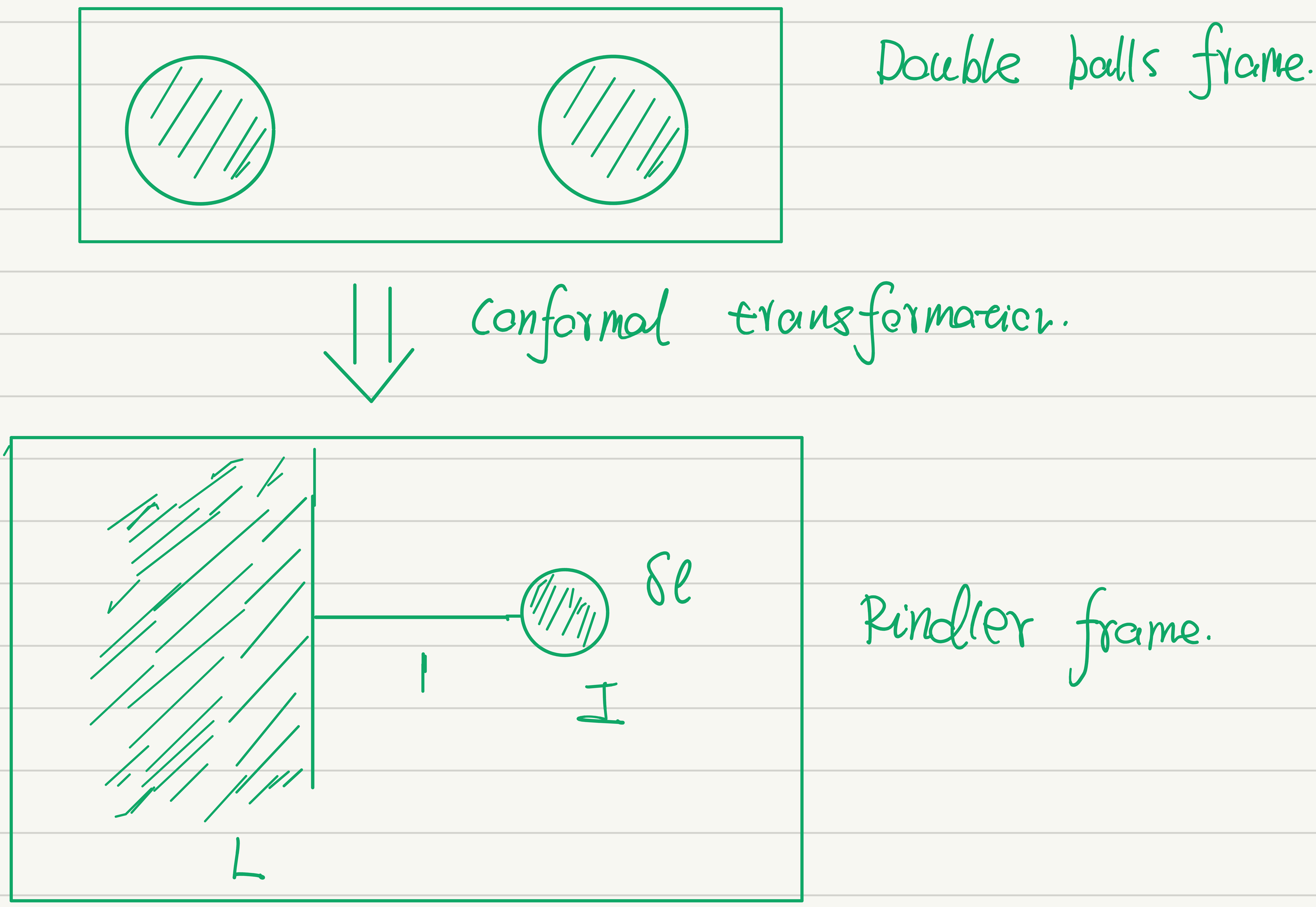
the OPE coefficients C_{j-k} can be extracted as.

$$\boxed{\begin{aligned} C_{j-k} &= \langle \mathcal{O}(\tau_j, 0)\, \mathcal{O}(\tau_k, 0) \rangle_{\mathbb{H}, n} \\ &= G_n(\tau_i - \tau_j), \quad \tau_i = (2i+1)\pi . \end{aligned}}$$

G_n is the Green function in hyperbolic space $\mathbb{H}$ at temperature $2\pi n$.

1.2 The Rindler frame.

It turns out the Rindler frame is convenient, thus we introduce it firstly.



Then, the mutual info is given by

$$\boxed{\begin{aligned} I(L,I) &= \lim_{n\to 1} \frac{1}{1-n}\left(\sum_{j\neq k} C_{j,k}^2\right)(\delta l)^{2a} \\ &= (\delta l)^{2a}\,\frac{\sqrt{\pi}}{4^{2a+1}}\,\frac{\Gamma(2a+1)}{\Gamma(2a+3/2)} \end{aligned}}$$

1.3 The moduler Hamiltonian.

By using

$$\boxed{\langle \varphi | H_{L\cup I} | \psi \rangle = -\partial_n \big|_{n=1} \mathrm{Tr}\left(\psi_L \, \rho_{L\cup I} \, \varphi_L^{\dagger} \, \rho_{L\cup I}^{\,n-1} \right)}$$

We can obtain the opz limit form of $H_{L\cup I}$.

$$\Delta H_{L\cup I} = -(\delta\ell)^{2a} \int_{-\infty}^{+\infty} ds \, k_1(s) \, \mathcal{O}_L(-is) \, \mathcal{O}_I$$

$$+ \frac{i(\delta\ell)^{2a}}{2\pi} \int_{-\infty}^{+\infty} ds_j \, ds_k \, k_2(s_j, s_k) \, \mathcal{O}_L(-is_k - is_j) \, \mathcal{O}_L(-is_j)$$

where.

$$k_1(s) = \frac{1}{4\cosh^2 s/2} \, C_1(-is + \pi)$$

$$k_2(s_j, s_k) = \frac{C_1(-is_k + \epsilon)}{4\cosh^2 s_j/2} \left(\frac{1}{e^{s_k + i\epsilon} - 1} + \frac{1}{e^{s_j + s_k} + 1} \right)$$

and $C_1(-is)$ is the analytical continuation of C_{j-k} at $n=1$.

$$C(-is) = \left(-4\sinh^2(s/2) \right)^{-\Delta}$$

For more details about the derivation, see

ref: arXiv: 2108.01093. (Faulkner etc. , 2021)

check the modular Hamiltonian $\Delta H_{L\cup I}$.

$$I(L,I) = \langle \Delta H_{L\cup I} \rangle$$

$$= (\delta e)^{2a} \frac{N\pi}{4^{2a+1}} \frac{\Gamma(2a+1)}{\Gamma(2a+3/2)}$$

---

2. Shape dependence of mutual info.

2.1 Rindler frame.

The deformation of the geometry : $x^\mu \to x^\mu + \xi^\mu$

via diffeomorphism equivalence.

$\Rightarrow$ the deformation of metric $\quad g_{\mu\nu} \to \tilde{g}_{\mu\nu}$.

Then, assuming $\xi^u(x)$ is small., the change of reduced density matrix to leading order in $\xi(x)$ is.

$$\boxed{\delta\rho_L = U^{\dagger}\circ\rho_L U - \rho_L = \frac{1}{2}\int \delta g^{uv}\rho_L \hat{T}_{uv} + \mathcal{O}(\xi^2)}$$

Thus, the change of mutual info is given by.

$$\begin{aligned}\delta I(L,I) &= \int_{\mathcal{M}} d^d x\, \partial^u \xi(x) \langle T_{uv}(x)\, \Delta H_{LUI} \rangle. \\ &= \int_{\partial M} d^{d-1} v \sqrt{h(v)}\, n^u \xi^u \langle T^{uv}(v)\, \Delta H_{LUI} \rangle.\end{aligned}$$

Let $z = x^1 + i x^0$, $\bar{z} = x^1 - i x^0$., then

$$\delta I(L,I) = (i) \int d^{d-1} x^i\, \xi(x^i) \oint_{z=0} dz \, \langle T_{zz}\, \Delta H_{LUI} \rangle + \langle T_{\bar{z}\bar{z}}\, \Delta H_{LUI} \rangle + h.c.$$

Thus, the main technical obstacle is to solve the above integral.

Let us focus on the integral.

$$\boxed{K(X^i) = \oint_{z=0} dz \left\langle \left(T_{zz}(X) + T_{z\bar{z}}(X) \right) \Delta H_{LUI} \right\rangle + h.c.}$$

Based on the explicit expression of modular Hamiltonian ΔH_{LUI}, we divide the above integral into two part, that is

Single fold integral part:

$$\boxed{SI \propto \oint_{z=0} dz \int ds\, k_1(s) \Big(\langle T_{zz}\, \mathcal{O}_L(-is)\, \mathcal{O}_I \rangle + \langle T_{z\bar{z}}\, \mathcal{O}_L(-is)\, \mathcal{O}_I \rangle \Big). \quad + h.c.}$$

Double fold integral part:

$$\boxed{DI \propto \oint_{z=0} dz \int ds_j\, ds_k\, k_2(s_j, s_k) \Big(\langle T_{zz}\, \mathcal{O}_L(-is_j - is_k)\, \mathcal{O}_L(-is_j) \rangle + \langle T_{z\bar{z}}\, \mathcal{O}_L(-is_j - is_k)\, \mathcal{O}_L(-is_j) \rangle \Big) + h.c.}$$

For a d dimensional CFT, the correlation function of stress tensor $T_{\mu\nu}$ and two primary scalar fields $\mathcal{O}$ is given by:

$$\boxed{\begin{aligned} &\langle \mathcal{O}(x_1)\,\mathcal{O}(x_2)\,T^{\mu\nu}(x_3)\rangle = \frac{C_{12T}\,H^{\mu\nu}(x_1,x_2,x_3)}{|x_{12}|^{2\Delta-d+2}\,|x_{13}|^{d-2}\,|x_{23}|^{d-2}} \\ &\text{with } \begin{cases} H^{\mu\nu} = V^\mu V^\nu - \frac{1}{d} V^2 \delta^{\mu\nu} \\ V^\mu = x_{13}^\mu / x_{13}^2 - x_{23}^\mu / x_{23}^2 \end{cases} \end{aligned}}$$

From this formula, we can work out

$$\langle T_{zz}\, \mathcal{O}_L(-is)\, \mathcal{O}_I \rangle,\quad \langle T_{z\bar{z}}\, \mathcal{O}_L(-is)\, \mathcal{O}_I \rangle$$

$$\langle T_{zz}\, \mathcal{O}_L(-is_j - is_k)\, \mathcal{O}_L(-is_j) \rangle$$

$$\langle T_{z\bar{z}}\, \mathcal{O}_L(-is_j - is_k)\, \mathcal{O}_L(-is_j) \rangle$$

which are very complicated. And it seems hopeless to solve the integrals which relate with them.

Fortunately, we found a rule to handle these integrals, which turns out to be very efficient.

Notice that the key is to extract the simple pole of these integrands, i.e.

$$\boxed{\oint_{z=0} dz\,(\cdots\cdots)\quad,\quad \oint_{\bar{z}=0} d\bar{z}\,(\cdots\cdots).}$$

However, the interior integrands have not simple poles, these poles can only emerge after we finish the integrals of modular parameters s, s_j and s_k.

There is a basic principle which guides us to extract these poles. That is:

Before taking the integrals of modular flow parameters, we impose the z and $\bar{z}$ equal zero in the interior integrands. If the integrals of modular flow parameters are finite, it turns out these integrals do not emerge poles thus, we can exchange the order of them and the contour integrals of z, $\bar{z}$.

If these integrals are divergent, it implies the poles will emerge due to these integrals. And we can analyse the asymptotic behavior of these integrands, thus concluding the order of poles

---

With the guidance of the above principle, we found:

① The integral of modular flow parameter s in the single fold integral part SI does not emerge the poles of z and $\bar{z}$. Thus $SI = 0$.

② In the double fold integral part, the integrals of S_j do contribute the poles. However the integrals of S_k do not. Thus we can perform the integral of S_j firstly, then extract the simple pole, and leave the integral of S_k in the end.

For explicit, we give the main part of DI term here as an illusteration.

It turns out that:

$$\boxed{\begin{aligned} DI \propto \oint dz \int_0^{+\infty} dv \frac{v^{2a}}{(v-1)^{4a+1-d}} \int_0^{+\infty} du \\ \times \frac{(u+1)^{-1}(uv+1)^{-1} u^{d+n}}{\left(uv+\frac{z}{1+D}\right)^{1+d/2}\left(u+\frac{z}{1+D}\right)^{1+d/2}} + h.c. \\ \text{where. } u = e^{S_i}, \quad v = e^{S_k}, \quad D = \sum_{i\geq 2} X_i^2 \end{aligned}}$$

For the u part of this integral, we can use Feynman parameter technique to combine the twe facter in the denominator., that is.

$$\boxed{\begin{aligned} &\int_0^{+\infty} du \frac{(u+1)^{-1}(uv+1)^{-1} u^{d+n}}{\left(uv+\frac{z}{1+D}\right)^{1+d/2}\left(u+\frac{z}{1+D}\right)^{1+d/2}} \\ &= \frac{\Gamma(2+d)}{\Gamma(1+d/2)^2} \int_0^1 dw\, w^{d/2}(1-w)^{d/2} \int_0^{+\infty} du \frac{(u+1)^{-1}(uv+1)^{-1}u^{d+n}}{\left(uvw+u(1-w)+\frac{z}{1+D}\right)^{2+d}} \end{aligned}}$$

Let $y = u + \delta$, $\delta = z/((1+D)(wv+1-w))$, we have.

$$
\begin{aligned}
&\int_0^{+\infty} du \, \frac{(u+1)^{-1}(uv+1)^{-1} u^{d+n}}{\left(u(uv+(1-w)) + z/(1+D)\right)^{d+2}} \\
&= \frac{1}{(wv+1-w)^{d+2}} \left(\int_\delta^{+\infty} dy \, y^{-(d+2)} (y-\delta)^{d+n} \right) (1 + \mathcal{O}(\delta)) \\
&= \frac{1}{(wv+(1-w))^{d+1+n}} \frac{\Gamma(1-n)\Gamma(d+1+n)}{\Gamma(d+2)} \left(\frac{z}{1+D}\right)^{-1+n} \left(1 + \mathcal{O}\left(\frac{z}{1+D}\right)\right)
\end{aligned}
$$

Finally, after taking the auxiliary integral w, we get.

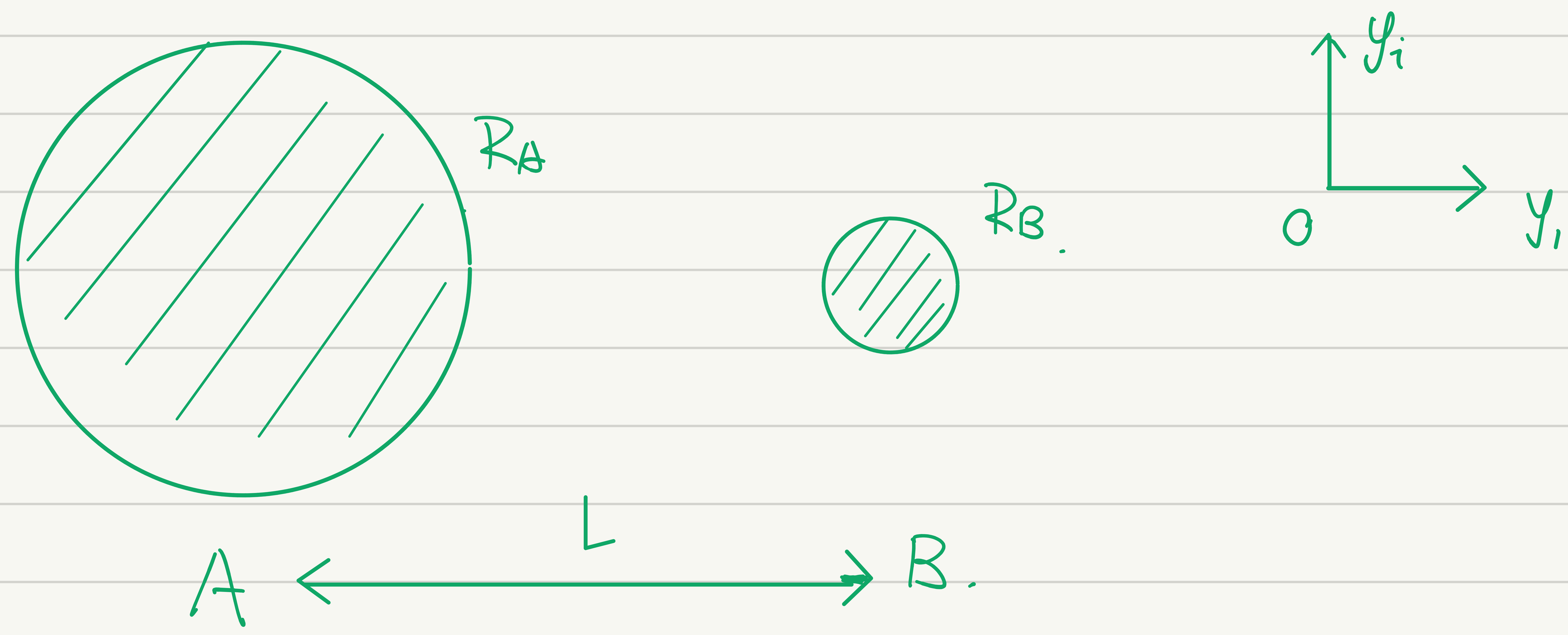
$$
\begin{aligned}
DI &\propto (1+D)^{-(d-1)} \int_0^{+\infty} dv \, \frac{v^{2a}}{(1-v)^{4a+1-d}} \, {}_2F_1(1+d, 1+d/2; 2+d; 1-v) \\
&\propto \Delta \frac{\sqrt{\pi}}{4^{2a+1}} \frac{\Gamma(2a+1)}{\Gamma(2a+3/2)} (1+D)^{-(d-1)}
\end{aligned}
$$

Combine all pieces together, we have

$$
\delta I_{L,I} = N_{\Delta,d} \int d^{d-2} X^i \, \xi(X^i) \left(1 + \sum_{i \geq 2} X_i^2\right)^{-(d-1)}
$$

$$
N_{\Delta,d} = \Delta (\delta\epsilon)^{2a} \frac{\sqrt{\pi}}{4^{2a+1}} \frac{\Gamma(2a+1)}{\Gamma(2a+3/2)} \frac{2^{d-1}\Gamma\left(\frac{d-1}{2}\right)}{\pi^{d/2-1/2}}
$$

2.2. Double balls frame.



The coordinate transformation is given by,

$$x^u = \frac{\tilde{y}^u + \tilde{y}^2 c^u}{1 + 2\,\tilde{y}\cdot c + c^2 \tilde{y}^2}$$

where.. $\tilde{y}^u = y^u - \left(R_A - L/2\right)\delta^u_1$, $\quad c^u = \delta^u_1 / 2R_A$

Let $\quad Y^u = y^u + L/2\,\delta^u_1$, $\quad X^u = G_2^{-1} x^u$,

To make the center of ball A at the origin, we do a shift.

$$Y^u = y^u + \frac{L}{2}\delta^u_1$$

In order to make the Rindler be standard, we also need a scaling transformation, $X^u = G_I^{-1} x^u$

(C_I is the new center of ball B)

then,

$$\delta I_{A,B} = \int \breve{\epsilon}\, d\theta \int d^{d-2}X_i\, \breve{n}^u \breve{\xi}^v \langle T_{uv}(X_i)\, \Delta H_{L,I} \rangle$$

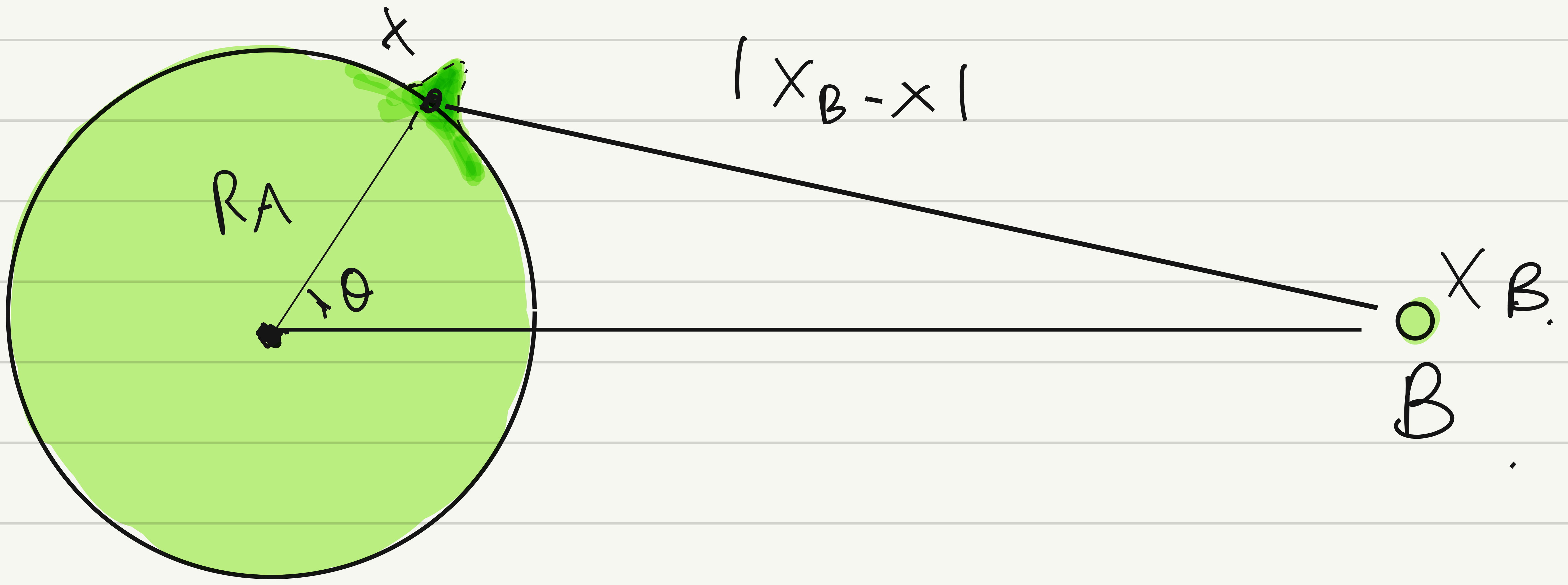
where.

$$\breve{\xi}^u = \left(1 + \frac{C_2^2}{4R_A^2} \sum_{i\geq 2} X_i^2\right) \breve{\zeta}(x) \;, \qquad \breve{\epsilon} = \epsilon \left(1 + \frac{C_2^2}{4R_A^2} \sum_{i\geq 2} X_i^2\right) \Big/ 2R_A$$

and $\breve{n}^u$ is the unit norm vector of Rindler frame.

In a word, we found.

$$\boxed{\begin{aligned} \delta I_{A,B} &= \# \int d\Omega_{d-2}\, \frac{\breve{\zeta}(\Omega_{d-2})}{\left(L^2 + R_A^2 - 2LR_A\cos\theta\right)^{d-1}} \\ \text{i.e.}\quad \frac{\delta I_{A,B}}{\sqrt{h(x)}\,\delta\zeta(x)} &\propto |x_B - x|^{-2(d-1)} \end{aligned}}$$



$$\boxed{\frac{\delta I_{A,B}}{\sqrt{h(x)}\,\delta\xi(x)} \propto |x_B - x|^{-(d-1)}}$$

Comments.

(i) the response density $\left(\frac{\delta I}{\delta \xi}\right)$ highly universal. i.e. independent of Δ for scalar O_Δ contribution.

(ii) does it depend on spin ℓ of $O_{\Delta,\ell}$?

— checked for OPE contribution from spin-1 primary : $\Sigma_B^{?} \to \sum_{ij} C_{ij}^{\mu\nu} \cdot O_\mu^i\, O_\nu^j$

— get the same factor. $\propto |x_B - x|^{-(d-1)}$

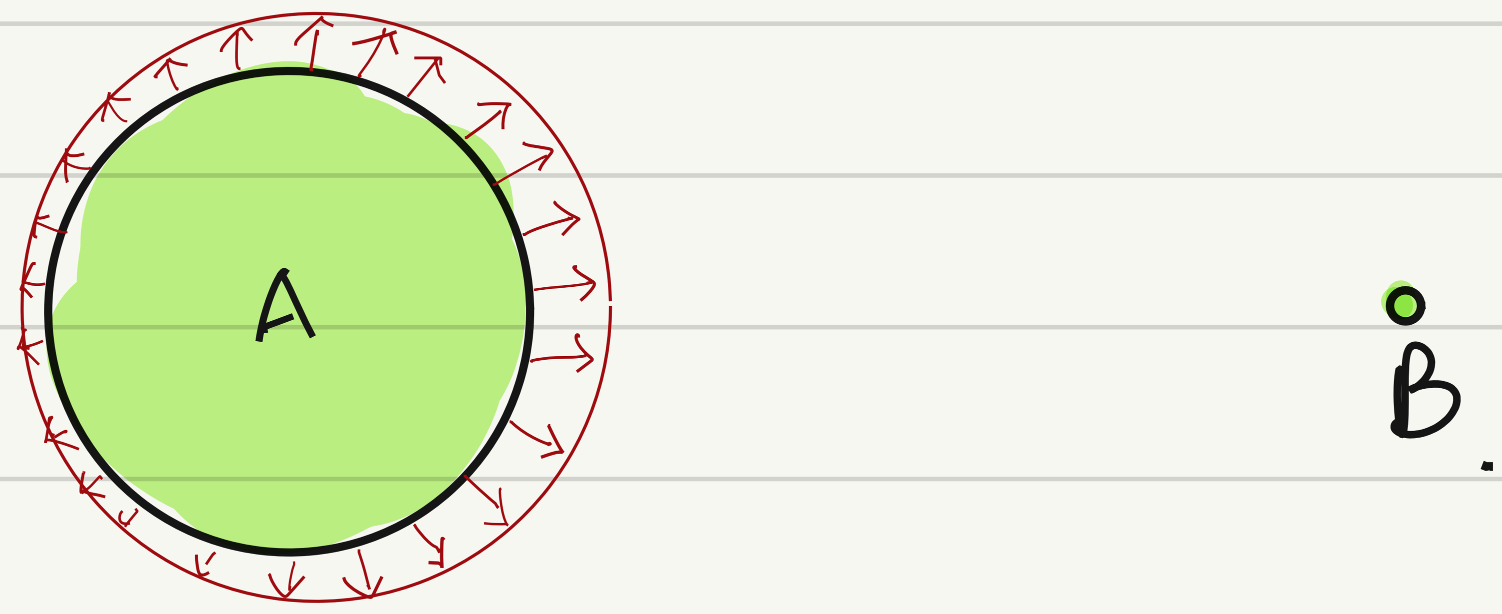
(iii) Conjecture :

$$\frac{\delta I_{A,B}}{\delta \xi(x)} \propto |x_B - x|^{-(d-1)} \quad \forall \text{ primary OPE}$$

contributions w/ arbitrary (Δ, ℓ).

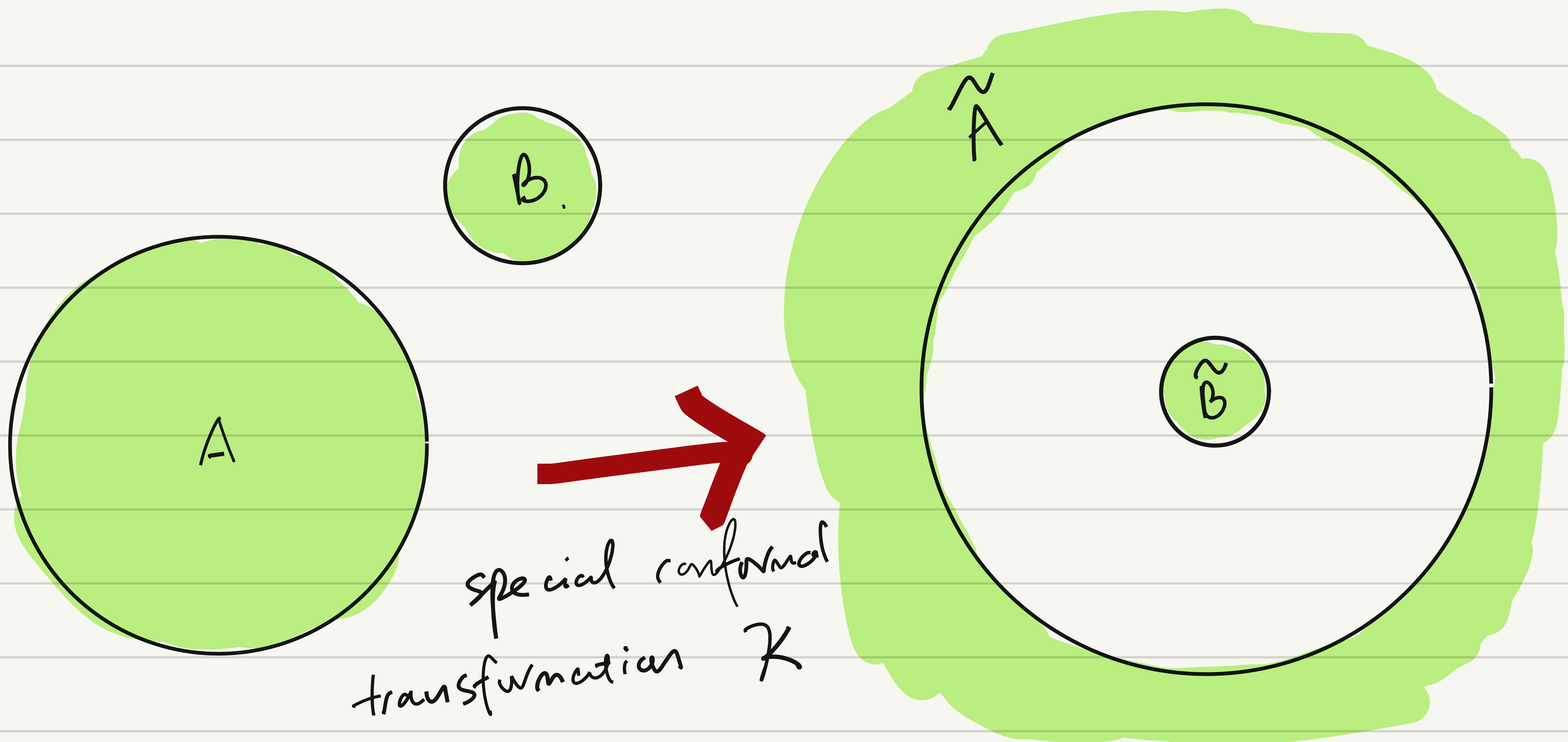
— Extremization of MI on spheres

$$\frac{\delta I_{A,B}}{\sqrt{h(x)}\,\xi(x)} \sim \frac{1}{|x_B - x|^{d-1}}$$



zero-mode on A? in what sense?

— For CFT vacuum $|\Omega\rangle$, $\exists$ symmetry group $\tilde{R}(\theta,\varphi)$ s.t. ① $\tilde{R}|\Omega\rangle = |\Omega\rangle$, ② leave A & B invariant



$$\therefore \; \tilde{R}(\theta,\varphi) = K^{-1}\, R(\theta,\varphi)\, K .$$

$\Uparrow$ ordinary rotation $SO(d-2)$

— Fourier analysis for $\tilde{R}$ on A

harmonic modes $\{\tilde{\Phi}_{l,m}\}$: $\tilde{\Phi}_{l,m} = \Phi_{l,m} \circ K$ (spherical harmonics)

Extremization of MI on spheres

(cont'ed).

inner products :

$$\langle \tilde{\Phi}_{\ell,m}, \tilde{\Phi}_{\ell',m'} \rangle = \int_{\partial_A} d\Omega^{d-2} \cdot J(\Omega) \cdot \tilde{\Phi}^*_{\ell,m}(\Omega) \tilde{\Phi}_{\ell',m'}(\Omega).$$

$$= \delta_{\ell m ; \ell' m'}$$

$$J(\Omega) \propto |x_B - x_\Omega|^{-(d-1)} = \text{our response density.}$$

$\therefore$ Interpretation of $\dfrac{\delta I}{\sqrt{h(x)}\,\xi(\Omega)} \propto |x_B - x_\Omega|^{-(d-1)} = J(\Omega)$:

$$\delta I(\xi) = \int d\Omega \cdot J(\Omega) \cdot \xi(\Omega) = \tilde{\xi}_{0,0}$$

— linear response only couple to zero-mode $\tilde{\xi}_{0,0}$ of the (conformal) rotation $\tilde{R}$ leaving A, B fixed.

— $\delta I(\xi_{\ell,m}) = 0$ for $(\ell,m) \neq (0,0)$.

$\therefore$ mutual information on spheres extremizes against non-zero modes of shape deformations $\xi(\ell,m)$.

Summary & Outlook

in this talk:

— computed linear response of MI between A & B spheres, under shape-deform. in the OPE limit i.e. $R_B \to 0$

— For scalar contribution, we found:

$$\frac{\delta I_{A,B}}{\sqrt{h(x)}\cdot\delta\xi(x)} \propto |x_B - x|^{-(d-1)}, \text{ independent } \Delta$$

— Checked for spin-$\ell=1$ case, response same

— Implication: $I_{A,B}$ extremizes on spheres over non-zero harmonic modes of $\tilde{R} = K^{-1} \circ R \circ K$.

In the future:

— Check further the universality: arbitrary spin ℓ; descendants; higher-pt op.

— higher-order response: $\frac{\delta^2 I}{\delta\xi\,\delta\xi} = ?$

— beyond OPE limit: re-sum R_B/ℓ corrections.

Thank You!