4d $\mathcal{N} = 2$ SCFT and modular differential equations

Yiwen Pan
based on 2104.12180, 2112.09705, 2207.10463
with Wolfger Peelaers; Yufan Wang, Haocong Zheng

August 25, 2022

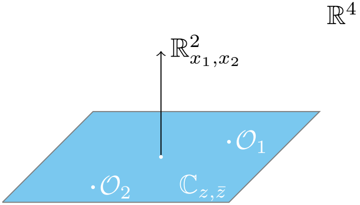
Outline

- Introduction to SCFT/VOA correspondence
- Closed-form Schur index
- flavored modular differential equations
- Modules characters from surface defects

Introduction

[Beem, Lemos, Liendo, Peelaers, Rastelli, Rees]

- 4d $\mathcal{N} = 2$ unitary SCFT $\mathcal{T}$ on $\mathbb{R}^4$



Associated VOA: a review

- SCFT/VOA correspondence: Schur operators form an associated **vertex operator algebra (VOA) $\mathbb{V}[\mathcal{T}]$**
- 4d $\mathcal{R}$-symmetry $\to$ Virasoro subalgebra $c_{\text{2d}} = -12 c_{\text{4d}} < 0$
- 4d $\mathfrak{f}$ flavor-symmetry $\to \widehat{\mathfrak{f}}_{k_{\text{2d}}}$ subalgebra, $k_{\text{2d}} = -\frac{1}{2} k_{\text{4d}}$

Associated VOA: a review

- Lots of existing literature on the subject
- identification of associated VOAs, VOA structure and modular differential equations, bounds, indices ...

  Class-$\mathcal{S}$ and T_N: [Beem, Peelaers, Rastelli, van Rees][Lemos, Peelaers][Kiyoshige, Nishinaka] ...

  Argyres-Douglas: [Song, Xie, Yan] [Xie, Yan, Yau] [Dedushenko, Wang] [Buican, Nishinaka] [Kozcaz, Shakirov, Yan][Creutzig] ...

  MDE, defects: [Cordova, Gaiotto, Shao][Bianchi, Lemos][Nishinaka, Sasa, Zhu][Beem, Rastelli][YP, Wang, Zheng] ...

  Free field realization: [Adamovic][Beem, Meneghelli, Rastelli][Bonetti, Meneghelli, Rastelli] ...

Associated VOA: Schur index

- Schur ops counted by the **Schur index** [Gadde, et.al.],

$$\mathcal{I}_{\mathcal{T}} \equiv \mathrm{str}_{\mathbb{V}[\mathcal{T}]}\, q^{E-R+\frac{c_{4d}}{2}} \mathbf{b}^{\mathbf{f}} = \mathrm{str}_{\mathbb{V}}\, q^{L_0 - \frac{c_{2d}}{24}} \mathbf{b}^{\mathbf{f}} \ , \qquad (1)$$

  where $q \equiv e^{2\pi i\tau}$, $\mathbf{b}, \mathbf{f}$ are flavor fugacities and Cartan gen's.
- Key equality: Schur index $\mathcal{I}_{\mathcal{T}}$ = vacuum character of $\mathbb{V}[\mathcal{T}]$

Computing Schur indices (focus on Lagrangian theories):

- Direct counting Schur opereators or identifying the VOA [Gadde, Rastelli, Razamat, Yan]: a series expansion
- From **2d q-Yang-Mills** partition functions [Gadde, Rastelli, Razamat, Yan]: an infinite sum over representations
- From **localization** on $S^3 \times S^1$, or zero-coupling limit (independence of g_{YM}) [Gadde, et.al.][YP, Peelaers][Dedushenko, Fluder][Jeong]: multivariate **contour integral** formula

$$\mathcal{I} = \oint_{|a|=1} \left[\frac{da}{2\pi i a}\right] \mathcal{Z}(a) \qquad (2)$$

Also compute **Schur correlators** on $S^3 \times S^1$ [YP, Peelaers]

Associated VOA: modules

VOAs are interesting objects from representation-theoretic perspective

- Rational VOA $\mathbb{V}$ (the "simplest" guys)
    - Finitely many irreducible $\mathbb{V}$-modules M_j
    - Any module M is a direct sum

$$M = \oplus_j M_j \ . \tag{3}$$

    - Modularity of characters ch_j and modular inv. partition function $Z \sim \sum_{i,j} \mathcal{M}_{ij}\, \mathrm{ch}_j\, \overline{\mathrm{ch}_j}$
    - Zero dimensional **associated variety $\mathcal{R}_{\mathbb{V}}$**
    - Other nice properties

Associated VOA: modules

- In SCFT/VOA correspondence: $\mathbb{V}[\mathcal{T}]$ is **non-unitary**, **non-rational** in general [Beem, et.al.]
- More complicated representation theory [Arakawa][Adamovic]
- Simplest module
  - Vacuum module $M_0 := \mathbb{V}$
  - Vacuum character $\mathrm{ch}_0 = \mathcal{I}_{\mathcal{T}}$
- Sources of **non-vacuum** modules from 4d physics: superconformal surface defects in $\mathcal{T}$ [Cordova, Gaiotto, Shao][Nishinaka, Sasa, Zhu][Bianchi, Lemos][Beem, Rastelli][Beem, Peelaers]

Goal

- Analytically compute Schur index and surface defect index in closed form
- Explore non-vacuum module characters for $\mathcal{T} = A_1$ theories of class-$\mathcal{S}$ theory in terms of surface defects

Exact Schur index in closed-form

Ellipticity

- Some convention: normal v.s. fraktur font

$$z = e^{2\pi i\mathfrak{z}}, \qquad y = e^{2\pi i\mathfrak{y}}, \qquad a = e^{2\pi i\mathfrak{a}}, \qquad b = e^{2\pi i\mathfrak{b}}$$

except

$$q = e^{2\pi i\tau} \ .$$

Ellipticity

- Lagrangian theory: Schur index as contour integral

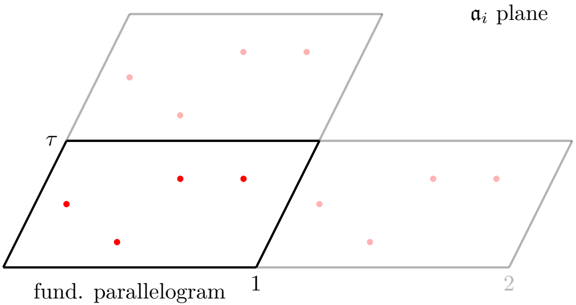
$$\mathcal{I} = \oint_{|a_i|=1} \prod_i \frac{da_i}{2\pi i a_i} \mathcal{Z}(a) \ . \tag{4}$$

- The integration contour is given by **unit circles** $|a_i| = 1$: **not** Jaffrey-Kirwan residue
- To get $\mathcal{I}$ in q-series: expand $\frac{1}{\prod_i a_i}\mathcal{Z}(a)$ in q-series, the integral computes residue at $a_i = 0$ of each term.

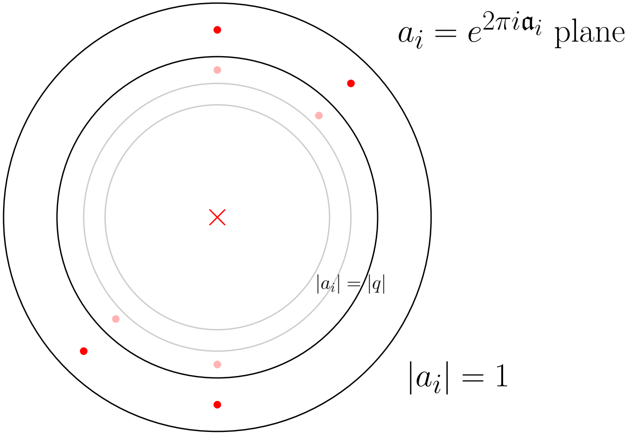
Ellipticity

- Integrand $\mathcal{Z}(a_i \equiv e^{2\pi i \mathfrak{a}_i})$ is **elliptic** w.r.t. **each** $\mathfrak{a}_i$ [Razamat]

$$\mathcal{Z}(\mathfrak{a}_i, \ldots) = \mathcal{Z}(\mathfrak{a}_i + 1, \ldots) = \mathcal{Z}(\mathfrak{a}_i + \tau, \ldots) \ , \forall i \ .$$



Ellipticity



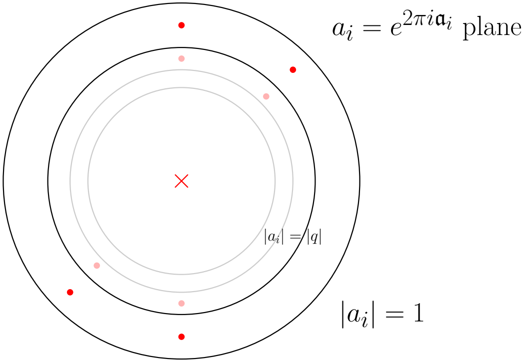
- Isolated singularities $\rightarrow$ **non-isolated** singularity $a_i = 0$

Ellipticity

- Fully flavored examples: isolated singularities are all **simple** poles with **residue**

$$R := \operatorname{Res}_{\mathsf{pole}} \frac{1}{a} \mathcal{Z}(a) \ . \tag{5}$$

Ellipticity



Problem:

- All isolated residues **cancel**: **trivial Residue theorem**
- **non-isolated** singularity at the origin $a = 0$: notion of residue **does not exist**.

Integrating Elliptic functions

- Various tricks involving Weierstrass ζ, $\wp$-function, Eisenstein series and Jacobi theta
- **First integral formula** (f has only simple poles): sum over poles in the **fundamental parallelogram**

$$\oint_{|a|=1} f(a)\frac{da}{2\pi i a} = f(a_0) \ + \sum_{\text{real/img. } \mathfrak{a}_i} R_i E_1 \begin{bmatrix} -1 \\ \frac{\mathfrak{a}_i}{a_0} q^{\pm\frac{1}{2}} \end{bmatrix} ,$$

a_0 is an arbitrary reference value

real/imaginary poles: $\operatorname{Im}\mathfrak{a}_i = 0$ **or** $\operatorname{Im}\mathfrak{a}_i > 0$.

Example: $\mathcal{N}=4$ $SU(2)$ theory

- $\mathcal{T} : \mathcal{N}=4$ $SU(2)$ SYM
- **$\mathbb{V}[\mathcal{T}] =$ 2d small $\mathcal{N}=4$ SCFA**
- The Schur index is a contour integral

$$\mathcal{I}_{\mathcal{N}=4}(b) = -\frac{1}{2}\oint \frac{da}{2\pi i a}\prod_{\pm}\frac{\vartheta_1(\pm\mathfrak{a})\eta(\tau)^3}{\vartheta_4(\pm\mathfrak{a}+\mathfrak{b})\vartheta_4(\mathfrak{b})} \quad .$$

- Two (imaginary) poles $a = b^{\pm 1} q^{\frac{1}{2}}$ with almost identical residues,

$$R_{\pm} = \pm\frac{i}{2}\frac{\vartheta_4(\mathfrak{b})}{\vartheta_1(2\mathfrak{b})} \quad \Rightarrow \quad \mathcal{I}_{\mathcal{N}=4} = \frac{i\vartheta_4(\mathfrak{b})}{\vartheta_1(2\mathfrak{b})} E_1\begin{bmatrix}-1\\ b\end{bmatrix} \ . \qquad (6)$$

Example: $\mathcal{N}=4$ $SU(2)$ theory

- The coefficient $\mathcal{I}_{bc\beta\gamma} = i\vartheta_4/\vartheta_1$ in front of E_1 is important:
  - The **residue** of the integrand
  - Precisely = vacuum character of $\mathbb{V}_{bc\beta\gamma}$ [YP, Wang, Zheng]

	L_0	J_0
(b,c)	$(\frac{3}{2},-\frac{1}{3})$	$(\frac{1}{2},-\frac{1}{2})$
(β,γ)	$(1,0)$	$(1,-1)$

  - $\mathbb{V}_{bc\beta\gamma}$ provides a **free field realization** of $\mathbb{V}[\mathcal{T}]$ [Bonetti, Meneghelli, Rastelli]
  - $\mathbb{V}_{bc\beta\gamma}$ is a reducible but **indecomposable** module of $\mathbb{V}[\mathcal{T}]$: **non-rationality**

Example: $\mathcal{N} = 4$ $SU(2)$ theory

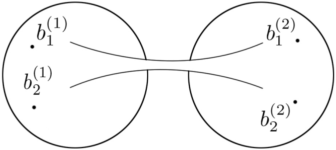
- The coefficient $\mathcal{I}_{bc\beta\gamma} = i\vartheta_4/\vartheta_1$ in front of E_1 is important:
  - Defect index of **Gukov-Witten type** surface defect [Gukov, Witten]
  - $\mathcal{I}_{g,n} = Z^{S^3_{\varphi,\chi,\theta} \times S^1_t}$ localized onto a $T^2_{\theta=0}$

$$\oint_{|a|=1} \frac{da}{2\pi i a} \mathcal{Z}(a) \tag{7}$$

  - Variables $a = e^{2\pi i \mathfrak{a}}$ encode flat gauge connection $A = \mathfrak{a} dt$
  - Gukov-Witten defect: singular profile $A \sim \mathfrak{a}_\varphi d\varphi$, singular field strength F peaks at $T^2_{\theta=\frac{\pi}{2}}$
  - Effect: $\mathfrak{a} \to \mathfrak{a} + \mathfrak{a}_\varphi \tau$, integration contour shifts $|a| = 1 \to |a| = |q^{-\mathfrak{a}_\varphi}|$.
  - Contour crosses pole: pick up residue $\mathcal{I}_{bc\beta\gamma}$.

Example: $SU(2)$ SQCD

- $SU(2)$ gauge theory with four fundamental hypers: the associated VOA is $\widehat{\mathfrak{so}}(8)_{-2}$



- Schur index $\mathcal{I}_{0,4}(b)$

$$= -\frac{1}{2}\oint \frac{da}{2\pi i a}\vartheta_1(\pm 2\mathfrak{a})^2 \prod_{j=1}^{4} \frac{\eta(\tau)}{\vartheta_4(\pm\mathfrak{a}+\mathfrak{m}_j)} \tag{8}$$

Example: $SU(2)$ SQCD

- 8 imaginary poles $a = m_j q^{\frac{1}{2}}$, 4 different **residues R_j**

$$R_j = \frac{i}{2} \frac{\vartheta_1(2\mathfrak{m}_j)}{\eta(\tau)} \prod_{\ell \neq j} \frac{\eta(\tau)}{\vartheta_1(\mathfrak{m}_j + \mathfrak{m}_\ell)} \frac{\eta(\tau)}{\vartheta_1(\mathfrak{m}_j - \mathfrak{m}_\ell)} . \qquad (9)$$

- R_j are index of **Gukov-Witten type defects**
- Apply the integral formula

$$\mathcal{I}_{0,4} = \sum_{j=1}^{4} \frac{i\vartheta_1(2\mathfrak{m}_j)}{\eta(\tau)} \prod_{\ell \neq j} \frac{\eta(\tau)}{\vartheta_1(\mathfrak{m}_j + \mathfrak{m}_\ell)} \frac{\eta(\tau)}{\vartheta_1(\mathfrak{m}_j - \mathfrak{m}_\ell)} E_1 \begin{bmatrix} -1 \\ m_j \end{bmatrix} .$$

Example: $SU(2)$ SQCD

- Parameters m's recombine into fugacities associated to the four punctures

$$m_1 = b_1^{(1)} b_2^{(1)}, \quad m_2 = \frac{b_1^{(1)}}{b_2^{(1)}}, \quad m_3 = b_1^{(2)} b_2^{(2)}, \quad m_4 = \frac{b_1^{(2)}}{b_2^{(2)}} .$$

  Manifest permutation invariance among $b_a^{(i)}$ is lost

- We will derive alternative and more elegant expression

General integral formula

- Higher ranks:

$$\oint \cdots \frac{da_2}{2\pi i a_2} \frac{da_1}{2\pi i a_1} \underbrace{\mathcal{Z}(a_1, \ldots, a_n)}_{\text{individually elliptic}} \tag{10}$$

- Problem: ellipticity is lost as function of $\mathfrak{a}_{2,\ldots,n}$

$$\oint \cdots \frac{da_2}{2\pi i a_2} \underbrace{\oint \frac{da_1}{2\pi i a_1} \mathcal{Z}(a_1, \ldots, a_n)}_{\text{non-elliptic in } \mathfrak{a}_{2,3,\ldots}} \tag{11}$$

$\mathfrak{a}_1$-integral contains **Eisenstein series** with variables $\mathfrak{a}_{2,3,\ldots}$

General integral formula

- **Integral formula** (with one or two E_k): e.g.,

$$\oint_{|a|=1} \frac{da}{2\pi i a} f(a) E_k \begin{bmatrix} -1 \\ ab \end{bmatrix}$$
$$= -\mathcal{S}_k \left(f(a_0) + \sum_{\text{real/img. } a_i} R_i E_1 \begin{bmatrix} -1 \\ \frac{a_i}{a_0} q^{\pm\frac{1}{2}} \end{bmatrix} \right)$$
$$- \sum_{\text{real/img. } a_i} R_i \sum_{\ell=0}^{k-1} \mathcal{S}_\ell E_{k-\ell+1} \begin{bmatrix} 1 \\ a_i b q^{\pm\frac{1}{2}} \end{bmatrix} , \tag{12}$$

where $\mathcal{S}_k$ is defined as

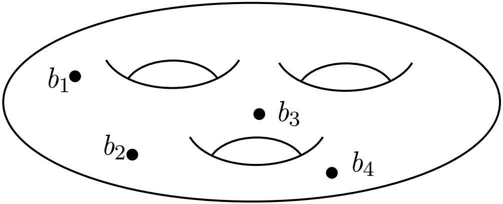
$$\frac{1}{2} \frac{y}{\sinh \frac{y}{2}} = \sum_{\ell \geq 0} \mathcal{S}_\ell y^\ell . \tag{13}$$

Higher-rank Lagrangian examples

- Compact formula for all A_1-theories of class-$\mathcal{S}$
- $SU(N)$ with $2N$ flavors (computable, compact formula not available yet)
- $\mathcal{N} = 4$ $G = SU(3), SU(4) = SO(6), SO(4), SO(5), SO(7)$ SYM

Examples: A_1 theories of class-$\mathcal{S}$

- A_1 theories of class-$\mathcal{S}$: specified by an n-punctured genus-g Riemann surface $\Sigma_{g,n}$



- 4d $\mathcal{N} = 2$ SCFT
- Lagrangian: $SU(2)$ vector multiplet + hypermultiplets
- Schur index $\mathcal{I}_{g,n}$

Examples: A_1 theories of class-$\mathcal{S}$

- Result for all $\mathcal{I}_{g,n}$

$$\mathcal{I}_{g,n}(b) = \frac{i^n}{2}\frac{\eta(\tau)^{n+2g-2}}{\prod_{i=1}^{n}\vartheta_1(2\mathfrak{b}_i)} \times \sum_{k=1}^{n+2g-2} \lambda_k^{(n+2g-2)} \sum_{\alpha=\pm}\Big(\prod_{i=1}^{n}\alpha_i\Big)E_k\left[\frac{(-1)^n}{\prod_{i=1}^{n} b_i^{\alpha_i}}\right]$$

$$\mathcal{I}_{g,n=0} = \frac{1}{2}\eta(\tau)^{2g-2} \times \sum_{k=1}^{g-1}\lambda_{2k}^{(2g-2)}\left(E_{2k} + \frac{B_{2k}}{(2k)!}\right)$$

- λ's are rational numbers: recursion relations

$$\lambda_0^{(\text{even})} = \lambda_{\text{even}}^{(\text{odd})} = \lambda_{\text{odd}}^{(\text{even})} = 0, \qquad \lambda_2^{(2)} = 1\ ,$$

$$\lambda_{2m+1}^{(2k+1)} = \sum_{\ell=m}^{k}\lambda_{2\ell}^{(2k)}\mathcal{S}_{2(\ell-m)}, \quad \lambda_{2m+2}^{(2k+2)} = \sum_{\ell=m}^{k}\lambda_{2\ell+1}^{(2k+1)}\mathcal{S}_{2(\ell-m)},$$

$$\lambda_1^{(2k+1)} = \sum_{\ell=1}^{k}\lambda_{2\ell}^{2k}\left(\frac{B_{2\ell}}{(2\ell)!} - \mathcal{S}_{2\ell}\right) \quad .$$

Examples: A_1 theories of class-$\mathcal{S}$

- Proof by recursion
- Input 1: three-punctured sphere [Gadde, Rastelli, Razamat, Yan]

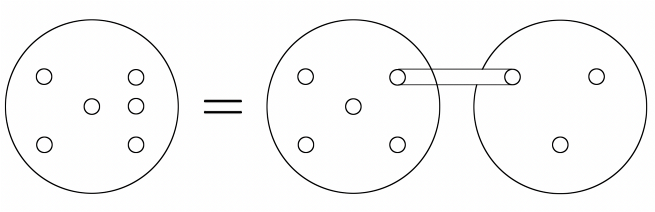
$$\mathcal{I}_{0,3} = \frac{1}{2i}\frac{\eta(\tau)}{\prod_{i=1}^{3}\vartheta_1(2\mathfrak{b}_i)}\sum_{\vec{\alpha}=\pm}\Big(\prod_{i=1}^{3}\alpha_i\Big)E_1\begin{bmatrix}(-1)^3\\ \prod_{i=1}^{3}b_i^{\alpha_i}\end{bmatrix}\ . \tag{14}$$

- Input 2: handle/vector multiplet

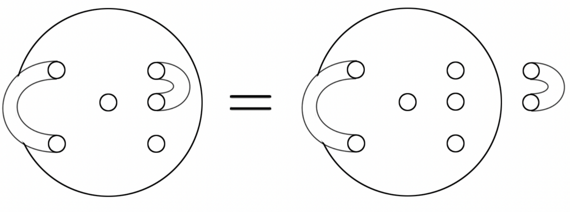
$$\oint\frac{da}{2\pi i a}\frac{1}{2}\vartheta_1(2\mathfrak{a})^2 \tag{15}$$

Examples: A_1 theories of class-$\mathcal{S}$

- $\mathcal{I}_{g,n+1}$ = gauging $\mathcal{I}_{g,n}$ and $\mathcal{I}_{0,3}$



- $\mathcal{I}_{g+1,n-2}$ = gauging two punctures of $\mathcal{I}_{g,n}$



- Schematic structure of $\mathcal{I}_{g,n}$

$$\mathcal{I}_{g,n>0}(b) \sim \frac{1}{\prod_{i=1}^{n} \vartheta_1(2\mathfrak{b}_i)} \vec{\lambda}_{g,n} \cdot \vec{E}\begin{bmatrix} (-1)^n \\ \prod_{i=1}^{n} b_i^{\alpha_i} \end{bmatrix} \tag{16}$$

- Product of $\vartheta_1(2\mathfrak{b}_i)$: flavor affine currents of the $\widehat{\mathfrak{su}}(2)_{-2}$ subalgebras, adjoint under the flavor $\mathfrak{su}(2)$'s
- The Eisenstein series: multi-fundamental operators, $(-1)^n$ reflect their conformal weights

Other applications

- Help check duality: e.g., generalized S-duality of the genus-two theory
- Conjectural compact formula for unflavored index of $\mathcal{N} = 4$ $SU(N)$ theories [Huang's talk]
- Non-Lagrangian Schur index in closed-form: E_6, E_7 SCFTs, $\widehat{\Gamma}(G)$ theories [Closset, Giacomelli, Schafer-Nameki, Wang][Kang, Lawrie, Song]
- Vortex defect index in closed-form
- Modular properties of Schur index and defect index

Application: $\mathcal{N}=4$ unflavored index

- Anzatz based on low-rank computation
- Conjectural unflavored indices for $SU(N)$,

$$\mathcal{I}_{\mathcal{N}=4\ SU(2N+1)} = (-1)^N \tilde{\lambda}_2^{(2N+3)} + (-1)^N \sum_{k=1}^{N} \frac{\tilde{\lambda}_{2k+2}^{(2N+3)}(2)}{2k} \widetilde{\mathbb{E}}_{2k} \ ,$$

$$\mathcal{I}_{\mathcal{N}=4\ SU(2N)} = (-1)^N \sum_{k=1}^{N} \frac{(-1)^k \tilde{\lambda}_{2k+1}^{(2N+2)}(2)}{(2k)!} \left(\frac{1}{2\pi}\right)^{2k-1} \frac{\vartheta_4^{(2k)}(0)}{\vartheta_1'(0)} \ .$$

where (below $[1^{n_1} 2^{n_2} \ldots]$ denotes a **integer partition of k**)

$$\widetilde{\mathbb{E}}_{2k} = \sum_{\substack{\{n_p\} \\ \sum_{p\geq 1} p n_p = k}} \prod_{p\geq 1} \frac{1}{n_p!} \left(-\frac{1}{2p} E_{2p}\right)^{n_p} \ . \qquad (17)$$

Application: $\mathcal{N} = 4$ unflavored index

- The formula were elucidated by using modular anomaly equation in [Huang]
- The fully flavored $\mathcal{N} = 4$ $SU(N)$ index were computed exactly using Fermi-gas formulation [Hatsuda, Okazaki], generalizing the results in [Bourdier, Drukker, Felix]

Application: vortex surface defect

- Focus on A_1 theories $\mathcal{T}_{g,n}$
- Vortex defect with **vorticity k** $\sim$ **poles** $b_i = q^{\frac{k+1}{2}}$ of $\mathcal{I}_{g,n+1}$

  [Gaiotto, Rastelli, Razamat]
- Vortex defect index with vorticity k [Cordova, Gaiotto, Shao][Alday, et.al.][Nishinaka, Sasa, Zhu]

$$\mathcal{I}^{\text{defect}}_{g,n}(k) = q^{-\frac{(k+1)^2}{2}} \operatorname*{Res}_{b \to q^{\frac{k+1}{2}}} \frac{2\eta(\tau)^2}{b} \mathcal{I}_{g,n+1}(b) \ . \qquad (18)$$

Application: vortex surface defect

- $k = 0$: recovers Schur index **without** defect

$$\mathcal{I}^{\text{defect}}_{g,n}(k=0) = \mathcal{I}_{g,n} \ . \tag{19}$$

- $k > 1$ and $n > 0$, simple poles

$$\mathcal{I}^{\text{defect}}_{g,n}(k) \sim \frac{\eta(\tau)^{n+2g-2}}{\prod_{i=1}^{n}\vartheta_1(2\mathfrak{b}_i)} \sum_{\ell=1}^{n+2g-1} \tilde{\lambda}_\ell^{(n+2g-1)}(k) \sum_{\alpha_i} \left(\prod_{i=1}^{n}\alpha_i\right) E_\ell \begin{bmatrix} (-1)^{n+k} \\ b_1^{\alpha_1}\dots b_n^{\alpha_n} \end{bmatrix} . \tag{20}$$

- $\tilde{\lambda}$: some rationals, also appears in $\mathcal{N}=4$ $SU(N)$ unflavored index

Application: vortex surface defect

- $\mathcal{I}^{\text{defect}}_{g,n}(k)$ resembles $\mathcal{I}_{g,n}$: linear combination of

$$E_k \begin{bmatrix} (-1)^n \\ \prod b \end{bmatrix} \text{ when } k \text{ even,} \qquad E_k \begin{bmatrix} -(-1)^n \\ \prod b \end{bmatrix} \text{ when } k \text{ odd.}$$

- Observation: when $k =$ odd, multi-fundamentals are **spectral flowed** by **half-integer** units

Application: vortex surface defect

- $k > 1$ and $n = 0$, **double** poles

$$\mathcal{I}^{\text{defect}}_{g,0}(k) \sim \sum_{n=0}^{g-1} c_n(k) E_{2n} \qquad \text{when } k \text{ even },\tag{21}$$

$$\sim \sum_{n=0}^{g-1} c'_n(k) E_{2n} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \text{when } k \text{ odd },\ .\tag{22}$$

Here c_n, c'_n are some rational numbers.

Flavored modular differential equations

- Modular (w.r.t. $\Gamma \subset SL(2,\mathbb{Z})$) form f of **weight-k**

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\,. \tag{23}$$

- Example: polynomials of standard Eisenstein series E_4, E_6
- Example: **quasi-modular** E_2

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 E_2(\tau) - \frac{c(c\tau+d)}{2\pi i}\,. \tag{24}$$

- Generalization: **twisted Eisenstein series**

$$E_{k\geq 1}\begin{bmatrix} e^{2\pi i\lambda} \\ z \end{bmatrix}(\tau) := -\frac{B_k(\lambda)}{k!} \tag{25}$$

$$+\frac{1}{(k-1)!}\sum_{r\geq 0}{}' \frac{(r+\lambda)^{k-1} z^{-1} q^{r+\lambda}}{1-z^{-1}q^{r+\lambda}} \tag{26}$$

$$+\frac{(-1)^k}{(k-1)!}\sum_{r\geq 1}\frac{(r-\lambda)^{k-1} z q^{r-\lambda}}{1-zq^{r-\lambda}}\ . \tag{27}$$

where $q=e^{2\pi i\tau}$.

- **Quasi-Jacobi forms** of **weight-k**: e.g., $(\tau,\mathfrak{z})\xrightarrow{S}(-\frac{1}{\tau},\frac{\mathfrak{z}}{\tau})$

$$E_k\begin{bmatrix} +1 \\ +z \end{bmatrix} \xrightarrow{S} \left(\frac{1}{2\pi i}\right)^n \left[\left(\sum_{\ell\geq 0}\frac{1}{\ell!}(-\log z)^\ell y^\ell\right)\left(\sum_{\ell\geq 0}(\log q)^\ell y^\ell E_\ell\begin{bmatrix} +1 \\ z \end{bmatrix}\right)\right]_k$$

Modular forms and more

- Differential operator $D_q^{(k)}$: weight-0 $\rightarrow$ weight-$2k$

$$D_q^{(k)} := \Big(q\partial_q + (2k-2)E_2\Big) \circ \cdots \circ \Big(q\partial_q + 2E_2\Big) \circ \Big(q\partial_q\Big)$$

MDEs from null states

- Any null state $\mathcal{N}$ of a VOA $\mathbb{V}$ and a $\mathbb{V}$-module M

$$0 = \mathrm{tr}_M (-1)^F \mathcal{N} q^{L_0 - \frac{c}{24}} \mathbf{b}^{\mathbf{f}} \, . \qquad (28)$$

Here $\mathbf{b}^{\mathbf{f}} = \prod_i b_i^{f_i}$

- **Zhu's recursion formula** helps compute one-point function recursively.

MDEs from null states

- **Zhu's recursion formula** [Zhu][Krauel, Mason][Beem, Rastelli][Beem, Peelaers][YP, Wang, Zheng]: when $J_0 a = 0$

$$\begin{aligned} & \mathrm{str}_M\, o(a_{[-h_a]} b) x^{J_0} q^{L_0} \\ = \ & \mathrm{str}_M\, o(a_{[-h_a]}|0\rangle) o(b) x^{J_0} q^{L_0} \\ & + \sum_{n=1}^{+\infty} E_{2k} \begin{bmatrix} e^{2\pi i h_a} \\ 1 \end{bmatrix} \mathrm{str}_M\, o(a_{[-h_a+2k]} b) x^{J_0} q^{L_0} \ , \end{aligned} \tag{29}$$

when $J_0 a \neq 0$,

$$\mathrm{str}_M\ o(a_{[-h_a]} b) x^{J_0} q^{L_0} = \sum_{n=1}^{+\infty} E_n \begin{bmatrix} e^{2\pi i h_a} \\ x^Q \end{bmatrix} \mathrm{str}_M\, o(a_{[-h_a+n]} b) x^{J_0} q^{L_0} \ .$$

MDEs from null states

- Any null state $\mathcal{N}$ of a VOA $\mathbb{V}$

$$0 = \mathrm{tr}_M(-1)^F \mathcal{N} q^{L_0 - \frac{c}{24}} \mathbf{b}^{\mathbf{f}} \ . \tag{30}$$

Here $\mathbf{b}^{\mathbf{f}} = \prod_i b_i^{f_i}$

- **Zhu's recursion formula**: for some nice $\mathcal{N}$, the above turns into a **flavored modular differential equation** for any module character ch_M [Gaberdiel, Keller][Gaberdiel, Lang][Beem, Rastelli][Beem, Peelaers][YP, Wang, Zheng][Zheng, YP, Wang]

$$\mathcal{D}(D_q^{(k)}, D_{b_j})\, \mathrm{ch}_M = 0 \ , \tag{31}$$

where coefficients are **twisted Eisenstein series**.

MDEs from null states

- For any associated VOA $\mathbb{V}(\mathcal{T})$: at least one such null $\mathcal{N}$ is expected to exist [Beem, Rastelli].
- Stress tensor T of $\mathbb{V}(\mathcal{T})$ should be **nilpotent** up to a null state $\mathcal{N}_T$ and $\varphi \in C_2(\mathbb{V}(\mathcal{T}))$: $\exists k \in \mathbb{N}_{\geq 1}$

$$L_{-2}^k |0\rangle = \varphi + \mathcal{N}_T \ . \tag{32}$$

$\Rightarrow$ **Special null** state $\mathcal{N}_T$ associated with the **nilpotency**

- All module characters of $\mathbb{V}(\mathcal{T})$ satisfy some (unflavored) MDE from $\mathcal{N}_T$ [Beem, Rastelli][Kaidi, et.al.].
- They may be **other** nulls

Example: symplectic boson

- $\beta\gamma$ system with conformal weights $h_\beta = h_\gamma = \frac{1}{2}$: associated VOA of one hypermultiplet
- $\beta(z) = \sum_{n\in\mathbb{Z}-\frac{1}{2}} \beta_n z^{-n-\frac{1}{2}}$, $\gamma(z) = \sum_{n\in\mathbb{Z}-\frac{1}{2}} \gamma_n z^{-n-\frac{1}{2}}$,

$$\beta(z)\gamma(w) \sim \frac{1}{z-w} \ . \tag{33}$$

- $U(1)$ current and Stress tensor

$$J = (\gamma\beta), \qquad T = \frac{1}{2}\beta\partial\gamma - \frac{1}{2}\gamma\partial\beta \ . \tag{34}$$

- **Vacuum module** from $|0\rangle$ (annihilated by $\beta_{n>-\frac{1}{2}}, \gamma_{n>-\frac{1}{2}}$)

$$\mathrm{ch}_0 = \mathrm{tr}\, q^{L_0 - \frac{c}{24}} b^{J_0} = \frac{\eta(\tau)}{\vartheta_4(\mathfrak{b})} \ . \tag{35}$$

Example: symplectic boson

- Simplest nulls

$$J - (\gamma\beta) = 0, \qquad T - \frac{1}{2}(\beta\partial\gamma - \gamma\partial\beta) = 0\ . \qquad (36)$$

- Zhu's recursion $\Rightarrow$ flavored modular differential equations

$$\left(D_b + E_1\begin{bmatrix}-1\\ b\end{bmatrix}\right)\mathrm{ch}_0 = 0\ , \quad \left(D_q^{(1)} - E_2\begin{bmatrix}-1\\ b\end{bmatrix}\right)\mathrm{ch}_0 = 0\ .$$

- **Unique** solution/character ch_0
- There are other nulls: additional but redundant FMDEs.

Example: symplectic boson

- Consider **twisted sector** (spectral flowed by $\frac{1}{2}$ unit),

$$\beta(z) = \sum_{n\in\mathbb{Z}} \beta_n z^{-n-\frac{1}{2}}, \qquad \gamma(z) = \sum_{n\in\mathbb{Z}} \gamma_n z^{-n-\frac{1}{2}} \tag{37}$$

- The twisted module from **vacuum $|0\rangle_{\frac{1}{2}}$** (annihilated by $\beta_{n\geq 0}, \gamma_{n\geq 1}$),

$$\mathrm{ch}_{\frac{1}{2}} = -i\frac{\eta(\tau)}{\vartheta_1(\mathfrak{b})} \ . \tag{38}$$

- The same two nulls and Zhu's recursion,

$$\left(D_b + E_1\begin{bmatrix}+1\\ b\end{bmatrix}\right)\mathrm{ch}_{\frac{1}{2}} = 0 \ , \quad \left(D_q^{(1)} - E_2\begin{bmatrix}+1\\ b\end{bmatrix}\right)\mathrm{ch}_{\frac{1}{2}} = 0 \ .$$

Example: symplectic boson

- Untwisted to twisted

  $\Leftrightarrow$ the $-1 \to +1$ in Eisenstein

  $\Leftrightarrow$ conformal weight goes from $(h_\beta = h_\gamma = \frac{1}{2})$ to $(h_\beta = 0, h_\gamma = 1)$.
- **Unique** solution/character in the twisted sector

Example: symplectic boson

- Under $SL(2,\mathbb{Z})$ transformation, $E_n\left[\begin{smallmatrix}\pm 1\\ b\end{smallmatrix}\right]$ are quasi-Jacobi: equations are **not** covariant
- Remedy: **y-extension** $(y := e^{2\pi i\mathfrak{y}})$

$$\mathrm{ch} = y^{k=-\frac{1}{2}}\,\mathrm{tr}\, q^{L_0-\frac{c}{24}}\, b^{J_0} \ , \tag{39}$$

and

$$(\tau, \mathfrak{b}, \mathfrak{y}) \xrightarrow{S} (-\frac{1}{\tau}, \frac{\mathfrak{b}}{\tau}, \mathfrak{y} - \frac{\mathfrak{b}^2}{\tau}) \ . \tag{40}$$

- If there are multiple b_j, include corresponding level k_j

Example: symplectic boson

- Operator $D_q^{(n)}$ transforms under S

$$D_q^{(n)} \to \Big(\tau^2 \partial_{(2n-2)} + \tau \sum_i \mathfrak{b}_i D_{b_i} + \mathfrak{b}_i^2 k_i - (2n-2)\frac{\tau}{2\pi i}\Big) \circ \cdots \circ \Big(\tau^2 \partial_{(2)} + \tau \sum_i \mathfrak{b}_i D_{b_i} + \mathfrak{b}_i^2 k_i - 2\frac{\tau}{2\pi i}\Big) \circ \Big(\tau^2 \partial_{(0)} + \tau \sum_i \mathfrak{b}_i D_{b_i} + \mathfrak{b}_i^2 k_i\Big) \ ,$$

and

$$D_{b_i} := b_i \partial_{b_i} \to \tau D_{b_i} + 2\mathfrak{b}_i k_i \ . \tag{41}$$

Example: symplectic boson

- After y-extension: **almost covariance**

$$D_b + E_1 \begin{bmatrix} -1 \\ b \end{bmatrix} \xrightarrow{STS} (\tau - 1)\left(D_b + E_1 \begin{bmatrix} -1 \\ b \end{bmatrix}\right)$$

$$D_q^{(1)} - E_2 \begin{bmatrix} -1 \\ b \end{bmatrix} \xrightarrow{STS} (\tau - 1)\mathfrak{b}\left(D_b + E_1 \begin{bmatrix} -1 \\ b \end{bmatrix}\right) + \frac{(\tau-1)^2}{\tau}\left(\tau\left(D_q^{(1)} - E_2 \begin{bmatrix} -1 \\ b \end{bmatrix}\right) + \mathfrak{b}\left(D_b + E_1 \begin{bmatrix} -1 \\ b \end{bmatrix}\right)\right) .$$

$$D_b + E_1 \begin{bmatrix} +1 \\ b \end{bmatrix} \xrightarrow{S} \tau\left(D_b + E_1 \begin{bmatrix} +1 \\ b \end{bmatrix}\right)$$

$$D_q^{(1)} - E_2 \begin{bmatrix} +1 \\ b \end{bmatrix} \xrightarrow{S} \tau^2\left(D_q^{(1)} - E_2 \begin{bmatrix} +1 \\ b \end{bmatrix}\right) + \mathfrak{b}\tau\left(D_b + E_1 \begin{bmatrix} +1 \\ b \end{bmatrix}\right) .$$

- STS/S maps untwisted/twisted **solution** to another untwisted/twisted **solution**

Example: $SU(2)$ theory with 4 flavors

- The associated VOA is $\mathbb{V}(\mathcal{T}) = \widehat{\mathfrak{so}}(8)_{-2}$, $c = -14$
- This VOA is shown to have five irreducible non-logarithmic highest weight modules [Arakawa, Kawasetsu]
  - Vacuum module $\mathbb{V}(\mathcal{T})$
  - Four modules $M_{j=1,2,3,4}$ with highest weight $\lambda = w_j(\omega_1 + \omega_3 + \omega_4) - \rho$ with $w_j = 1, s_{1,3,4}$
- Can we recover them by analyzing FMDEs?

Example: $SU(2)$ theory with 4 flavors

- There are a lot of null states in $\widehat{\mathfrak{so}}(8)_{k=-2}$
  - $T - \frac{1}{2(k+h^\vee)} \sum_{a,b} K_{ab} J^a J^b = 0$
  - $(J^a J^b)_{\mathbf{35}_v,\mathbf{35}_s,\mathbf{35}_c} = 0$
  - Nulls at level 3, 4 ($\mathcal{N}_T$)
- 10 weight-two, 4 weight-three and 1 weight-four FMDEs for $\mathcal{I}_{0,4}$: e.g.,

$$
\begin{aligned}
0 = \Bigg[D_q^{(1)} & - \frac{1}{4}\left(D_{b_3}D_{b_2} + D_{b_4}D_{b_2} + D_{b_4}D_{b_3} + D_{b_4}^2\right) \\
& - \frac{1}{2}E_1\left[\begin{matrix} 1 \\ \frac{b_1}{b_2 b_3 b_4}\end{matrix}\right]\left(D_{b_1} - D_{b_2} - D_{b_3} - D_{b_4}\right) \\
& - \frac{1}{2}E_1\left[\begin{matrix} 1 \\ b_1 b_2 b_3 b_4\end{matrix}\right]\left(D_{b_1} + D_{b_2} + D_{b_3} + D_{b_4}\right) - E_1\left[\begin{matrix} 1 \\ b_4^2\end{matrix}\right] D_{b_4} \\
& + \left(E_2 + 2E_2\left[\begin{matrix} 1 \\ \frac{b_1}{b_2 b_3 b_4}\end{matrix}\right] + 2E_2\left[\begin{matrix} 1 \\ b_1 b_2 b_3 b_4\end{matrix}\right] + 2E_2\left[\begin{matrix} 1 \\ b_4^2\end{matrix}\right]\right)\Bigg]\mathcal{I}_{0,4} \ .
\end{aligned}
$$

Example: $SU(2)$ theory with 4 flavors

- Weight-three

$$0 = \Bigg[\sum_{i=1}^{4} c_i \left(D_q^{(1)} D_{b_i} + E_2 D_{b_i} - 2E_2 \begin{bmatrix} 1 \\ b_i^2 \end{bmatrix} D_{b_i} + 8E_3 \begin{bmatrix} 1 \\ b_i^2 \end{bmatrix} \right) - \frac{1}{4} \sum_{\alpha_i = \pm} E_2 \begin{bmatrix} 1 \\ \prod_{k=1}^4 b_k^{\alpha_k} \end{bmatrix} \sum_{i,j=1}^{4} \alpha_i \alpha_j c_i D_{b_j} + 2 \sum_{\alpha_i = \pm} \sum_{i=1}^{4} \alpha_i c_i E_3 \begin{bmatrix} 1 \\ \prod_{k=1}^4 b_k^{\alpha_k} \end{bmatrix} \Bigg] \mathcal{I}_{0,4} \; , \tag{42}$$

where c_i are arbitrary constants.

Example: $SU(2)$ theory with 4 flavors

- Weight-four (from $\mathcal{N}_T$)

$$0 = \Bigg(D_q^{(2)} + \frac{1}{2} \sum_{\alpha_i = \pm} E_3 \begin{bmatrix} +1 \\ \prod_{i=1}^4 b_i^{\alpha_i} \end{bmatrix} \Big(\sum_{i=1}^4 \alpha_i D_{b_i} \Big) + 2 \sum_{i=1}^4 E_3 \begin{bmatrix} +1 \\ b_i^2 \end{bmatrix} D_{b_i}$$
$$- 12 \sum_{i=1}^4 E_4 \begin{bmatrix} +1 \\ b_i^2 \end{bmatrix} - 31 E_4 - 6 \sum_{\alpha_i = \pm} E_4 \begin{bmatrix} +1 \\ \prod_i b_i^{\alpha_i} \end{bmatrix} \Bigg) \mathcal{I}_{0,4} \ .$$

- Unflavoring [Arakawa][Beem, Rastelli]

$$0 = (D_q^{(2)} - 175 E_4) \mathcal{I}_{0,4} \ . \tag{43}$$

- Note the origin of coefficients: $SU(2)_i$ adjoint moment maps or multi-fundamentals.

Example: $SU(2)$ theory with 4 flavors

- 15 equations have precisely 5 **non-logarithmic solutions** of the form [Beem, Peelaers]

$$q^{\alpha}\sum_{n=0}^{+\infty} a_n(b_1,\ldots,b_4)\,q^n\ , \tag{44}$$

given by

$$\mathcal{I}_{0,4}, \qquad R_{j=1,\ldots,4}\ . \tag{45}$$

- Correspond to the five highest weight modules (including the vacuum module) by Arakawa and Kawasetsu
- R_j do **not** have smooth $b_i \to 1$ limit: M_j do not have finite-dimensional weight spaces, **invisible** if unflavored

Example: $SU(2)$ theory with 4 flavors

- All the relevant equations are **almost covariant** under S (after **y-extension**):

$$S(\text{weight-2}) = \tau^2(\text{weight-2}) \tag{46}$$

$$S(\text{weight-3}) = \tau^3(\text{weight-3}) + \sum_{j=1}^{4} \tau^2 \mathfrak{b}_j(\text{weight} - 2)_j$$

$$\begin{aligned} S(\text{weight-4}) = \tau^4(\text{weight-4}) + \tau^3(\text{weight-3}) \\ + \tau^2 \sum_{ij} \mathfrak{b}_i \mathfrak{b}_j (\text{weight-2})_{ij} \ , \end{aligned} \tag{47}$$

Example: $SU(2)$ theory with 4 flavors

- Corollary: $S\mathcal{I}$ is also a **solution**, and is **logarithmic**

$$S\mathcal{I} = \frac{\log q}{2\pi} + \frac{1}{\pi}\sum_j (\log m_j) R_j \ . \qquad (48)$$

- $S^3 \times S^1$: $S\mathcal{I}$ can also be viewed as some **defect index**
- S, T matrix: $\mathrm{ch}_0 = \mathcal{I}, \mathrm{ch}_1 = S\mathcal{I}$

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} e^{\frac{7\pi i}{6}} & 0 \\ e^{-\frac{\pi i}{3}} & e^{-\frac{5\pi i}{6}} \end{pmatrix} \ . \qquad (49)$$

- Null $\mathcal{N}_T$ + modularity generates all the lower-weight FMDEs: determines all the allowed characters.

Example: $SU(2)$ theory with 4 flavors

- The above discussion concerns the untwisted sector
- Consider vortex defect index $\mathcal{I}^{\text{defect}}_{0,4}(k)$:

  Only $1 = \lceil \frac{n+2g-2}{2} \rceil$ independent index with $k =$ **even**:

$$\mathcal{I}^{\text{defect}}_{0,4}(k=0) = \mathcal{I}_{0,4} \ . \tag{50}$$

  Only $1 = \lceil \frac{n+2g-2}{2} \rceil$ independent index with $k =$ **odd**:

$$\mathcal{I}^{\text{defect}}_{0,4}(k=1) = \frac{\eta(\tau)^2}{\prod_{i=1}^{4} \vartheta_1(2\mathfrak{b}_i)} \sum_{\vec{\alpha}=\pm} \left(\prod_{i=1}^{4} \alpha_i \right) E_2 \left[\begin{matrix} -1 \\ \prod_{i=1}^{4} b_i^{\alpha_i} \end{matrix} \right] \ .$$

Example: $SU(2)$ theory with 4 flavors

- If $k = 1$ vortex defect corresponds to some module M_1, then

$$\mathcal{I}_{0,4}^{\text{defect}}(k=1) = \operatorname{str}_{M_1} q^{L_0 - \frac{c}{24}} \mathbf{b}^{\mathbf{f}} \ . \tag{51}$$

and

$$0 = \operatorname{str}_{M_1} \mathcal{N} q^{L_0 - \frac{c}{24}} \mathbf{b}^{\mathbf{f}} \ . \tag{52}$$

- From $\mathcal{I}_{0,4}^{\text{defect}}(k=1)$: all multi-fundamentals have half integral weights
- For all $\mathcal{N}$, Zhu's recursion still goes through:
  - FMDEs are **almost identical** to those for $\mathcal{I}_{0,4}$
  - all $E_n\begin{bmatrix} +1 \\ \prod_i b_i^{\alpha_i} \end{bmatrix}$ should be replaced by $E_n\begin{bmatrix} -1 \\ \prod_i b_i^{\alpha_i} \end{bmatrix}$

Example: $SU(2)$ theory with 4 flavors

- weight-two:

$$0 = \Bigg[D_q^{(1)} - \frac{1}{4}\left(D_{b_3} D_{b_2} + D_{b_4} D_{b_2} + D_{b_4} D_{b_3} + D_{b_4}^2\right) - \frac{1}{2} E_1 \begin{bmatrix} -1 \\ \frac{b_1}{b_2 b_3 b_4} \end{bmatrix} \left(D_{b_1} - D_{b_2} - D_{b_3} - D_{b_4}\right) - \frac{1}{2} E_1 \begin{bmatrix} -1 \\ b_1 b_2 b_3 b_4 \end{bmatrix} \left(D_{b_1} + D_{b_2} + D_{b_3} + D_{b_4}\right) - E_1 \begin{bmatrix} 1 \\ b_4^2 \end{bmatrix} D_{b_4} + \left(E_2 + 2E_2 \begin{bmatrix} -1 \\ \frac{b_1}{b_2 b_3 b_4} \end{bmatrix} + 2E_2 \begin{bmatrix} -1 \\ b_1 b_2 b_3 b_4 \end{bmatrix} + 2E_2 \begin{bmatrix} 1 \\ b_4^2 \end{bmatrix} \right) \Bigg] \mathcal{I}_{0,4}^{\text{defect}}(1) \ ,$$

Example: $SU(2)$ theory with 4 flavors

- weight-three:

$$0 = \left[\sum_{i=1}^{4} c_i \left(D_q^{(1)} D_{b_i} + E_2 D_{b_i} - 2E_2 \begin{bmatrix} 1 \\ b_i^2 \end{bmatrix} D_{b_i} + 8E_3 \begin{bmatrix} 1 \\ b_i^2 \end{bmatrix} \right) - \frac{1}{4} \sum_{\alpha_i = \pm} E_2 \begin{bmatrix} -1 \\ \prod_{k=1}^{4} b_k^{\alpha_k} \end{bmatrix} \sum_{i,j=1}^{4} \alpha_i \alpha_j c_i D_{b_j} + 2 \sum_{\alpha_i = \pm} \sum_{i=1}^{4} \alpha_i c_i E_3 \begin{bmatrix} -1 \\ \prod_{k=1}^{4} b_k^{\alpha_k} \end{bmatrix} \right] \mathcal{I}_{0,4} \,, \tag{53}$$

- True for all other FMDEs.
- Corollary: vortex defect with odd vorticity $\Leftrightarrow$ twisted modules
- Almost covariance: $STS\mathcal{I}_{0,4}^{\text{defect}}(k=1)$ is also a solution: **logarithmic twisted module**

Example: $\mathcal{N}=4$ $SU(2)$ theory

- The associated VOA $\mathbb{V}$ is 2d small $\mathcal{N}=4$ superconformal algebra with $c=-9$, contains a subalgebra $\widehat{\mathfrak{su}}(2)_{k=-\frac{3}{2}}$
- Schur index

$$\mathcal{I} = \frac{1}{2\pi}\frac{\vartheta_4'(\mathfrak{b})}{\vartheta_1(2b)} = \frac{i\vartheta_4(\mathfrak{b})}{\vartheta_1(2\mathfrak{b})} E_1 \begin{bmatrix} -1 \\ b \end{bmatrix} . \qquad (54)$$

- Nulls at dimension 2, 3, 4. E.g., the Sugawara construction

$$0 = T - \frac{1}{2(k+h^\vee)} \sum_{a,b} K_{ab} J^a J^b . \qquad (55)$$

And (schematically) [Beem, Rastelli]

$$\mathcal{N}_T \sim (L_{-2}^2 + \tilde{G}_{-\frac{5}{2}} G_{-\frac{3}{2}} + J_{-2}J_{-2} + L_{-4})|0\rangle . \qquad (56)$$

- three nulls of level-2, 3, 4 $\mathcal{N}_{2,3,4} \Rightarrow$ weight-2,3,4 FMDEs, e.g., the weight-2 equation is

$$\left[D_q^{(1)} - \frac{1}{2(k+h^\vee)}\left(\frac{1}{2}D_b^2 + kE_2 + 2kE_2\begin{bmatrix}1\\b^2\end{bmatrix} + 2E_1\begin{bmatrix}1\\b^2\end{bmatrix}D_b\right)\right]\mathcal{I}_{\mathcal{N}=4} = 0\ .$$

The weight-4 equation

$$\begin{aligned}0 = &(D_q^{(2)} + \frac{c_{\mathsf{2d}}}{2}E_4)\mathcal{I}_{\mathcal{N}=4}\\ &+\left(-2E_2\begin{bmatrix}-1\\b\end{bmatrix}D_q^{(1)} - 4E_3\begin{bmatrix}-1\\b\end{bmatrix}D_b + 18E_4\begin{bmatrix}-1\\b\end{bmatrix}\right)\mathcal{I}_{\mathcal{N}=4}\\ &+\left(3k_{\mathsf{2d}}E_4 + 2E_3\begin{bmatrix}1\\b^2\end{bmatrix}D_b - 9E_4\begin{bmatrix}1\\b^2\end{bmatrix}\right)\mathcal{I}_{\mathcal{N}=4}\ . \end{aligned} \tag{57}$$

Example: $\mathcal{N}=4$ $SU(2)$ theory

- Precisely 3 solutions ($\mathcal{I}_{bc\gamma\beta}$ is a residue)

$$\underbrace{\mathcal{I}, \qquad \mathcal{I}_{bc\beta\gamma},}_{\text{non-logarithmic}} \qquad \underbrace{STS\mathcal{I}}_{\text{logarithmic}} \quad . \tag{58}$$

- Modules of $\mathbb{V}$ were studied [Adamovic]: only two irreducible modules from category-$\mathcal{O}$

$$\mathbb{V}, \qquad \mathbb{V}_{bc\beta\gamma}/\mathbb{V} \ . \tag{59}$$

  Precisely correspond to the two non-Logarithmic solutions

- Flavoring is **necessary**: $\mathcal{I}_{bc\beta\gamma}$ **does not** have smooth $b \to 1$ limit, conformal-weight-spaces are infinite dimensional due to zero mode c_0

Example: $\mathcal{N}=4$ $SU(2)$ theory

- For an **unflavored** equation, the coefficients are STS **modular**: equation is **covariant** under STS

$$\text{solution} \xrightarrow{\text{modular trans}} \text{solution} \tag{60}$$

- **Question**: is $STS\mathcal{I}$ a solution when flavored?
- Almost covariance after **y-extension** with $k=-3/2$,

$$STS(\text{weight-2}) = (\tau-1)^2(\text{weight-2}) \ , \tag{61}$$

$$STS(\text{weight-3}) = (\tau-1)^3(\text{weight-3}) - 2\mathfrak{b}(\tau-1)^2(\text{weight-2}) \ ,$$

$$STS(\text{weight-4}) = (\tau-1)^4(\text{weight-4}) + 2\mathfrak{b}(\tau-1)^3(\text{weight-3}) - 2\mathfrak{b}^2(\tau-1)^2(\text{weight-2}) \ .$$

Example: $\mathcal{N} = 4$ $SU(2)$ theory

- FMDE from $\mathcal{N}_T$ + modularity generates all other FMDEs: determines all three characters.

Example: $\mathcal{N} = 4$ $SU(2)$ theory

- $STS\mathcal{I}$ is **logarithmic**,

$$STS\mathcal{I} = -y^{\frac{3}{2}} \frac{2\pi i \vartheta_4(\mathfrak{b}) + (\tau - 1)\vartheta_4'(\mathfrak{b})}{2\pi \vartheta_1(2\mathfrak{b})} \tag{62}$$

- Likely corresponds to some logarithmic module of $\mathbb{V}[\mathcal{T}]$

- Conjecture/observation: untwisted module characters

$$\text{Schur index } \mathcal{I}_{g,n} \tag{63}$$

$$\mathcal{I}_{g,n} = \oint \left[\frac{da}{2\pi i}\right] \underbrace{\frac{1}{a}\mathcal{Z}(a)}_{\text{residue/Gukov-Witten defect index}} \tag{64}$$

$$\mathcal{I}^{\text{defect}}_{g,n}(k = \text{even}) = \operatorname{Res} \frac{1}{a}\mathcal{I}_{g,n+1} \ . \tag{65}$$

- Conjecture/observation: **logarithmic** untwisted module characters

$$\text{modular trans. on } \mathcal{I}_{g,n},\ \mathcal{I}^{\text{defect}}_{g,n}(k = \text{even})\ . \qquad (66)$$

◇ Some conjectures

- Conjecture/observation: **twisted** module characters

$$\mathcal{I}^{\text{defect}}_{g,n}(k = \text{odd}) = \operatorname{Res} \frac{1}{a}\mathcal{I}_{g,n+1} \ . \tag{67}$$

There may be more.

◇ Some conjectures

- Conjecture/observation: **logarithmic** **twisted** module characters

$$\text{modular trans. on } \mathcal{I}^{\text{defect}}_{g,n}(k = \text{odd}) \tag{68}$$

Conclusion

- Propose some elementary method to compute Schur index and defect index exactly (for some theories)
- For some A_1 theories of class-$\mathcal{S}$: study (twisted) flavored modular differential equations of and their solutions
- These solutions originate from some surface defects

Thank you