第三届全国场论与弦论学术研讨会

形状因子研究新进展

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北京，2022年 8月23-26日

Outline

- Introduction to form factors
- Master-bootstrap and MTP
- CK-duality and double-copy

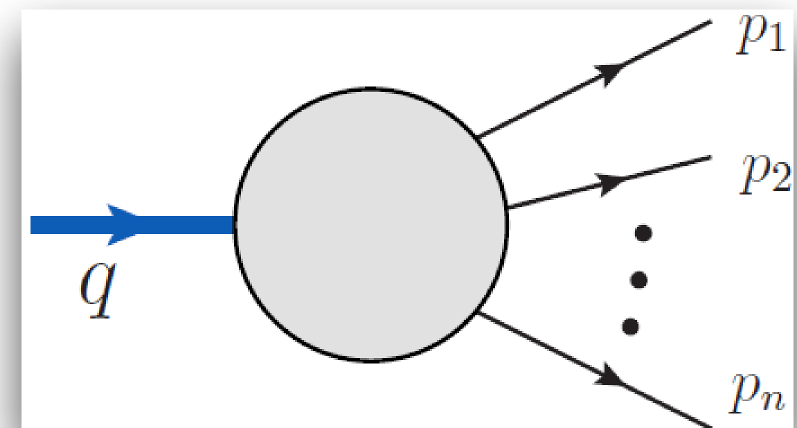
On-shell form factors

Hybrids of on-shell states and off-shell operators:

$$F_{n,\mathcal{O}}(1,\ldots,n) = \int d^4x\, e^{-iq\cdot x}\, \langle p_1 \ldots p_n | \mathcal{O}(x) | 0 \rangle$$

$$= \delta^{(4)}\Big(\sum_{i=1}^{n} p_i - q\Big)\, \langle p_1 \ldots p_n | \mathcal{O}(0) | 0 \rangle$$

(work in momentum space)



$$q = \sum_i p_i, \qquad q^2 \neq 0$$

$$\langle p_1 p_2 \ldots p_n | 0 \rangle$$

$$\langle \mathcal{O}_1 \mathcal{O}_2 \ldots \mathcal{O}_n \rangle$$

Operator examples

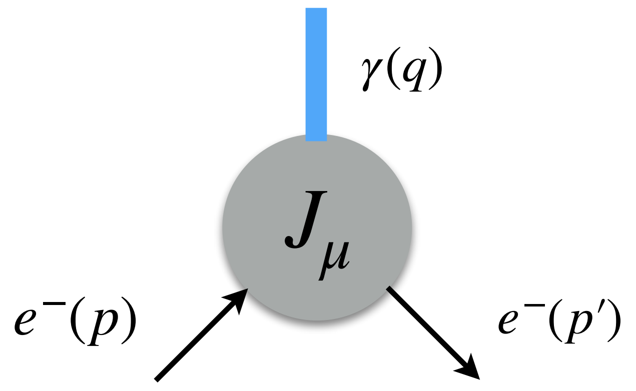
Operators are important quantities in QFT.

- Examples include conserved currents, such as stress-tensor $T_{\mu\nu}$, and U(1) current in QED J_μ

  - Electromagnetic form factor (**g-2**)

$$\langle e^-(p')\,|\,J_\mu(0)\,|\,e^-(p)\rangle =$$

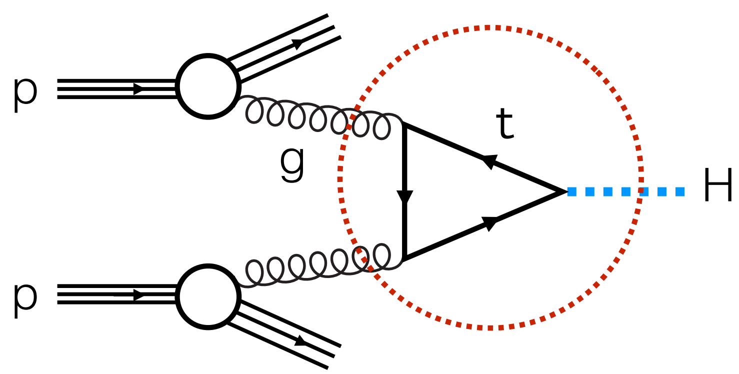
$$J_\mu = \bar{\psi}\gamma_\mu\psi$$



See e.g. Cvitanovic, Kinoshita 1974 (for a 3-loop computation!)

Operator examples

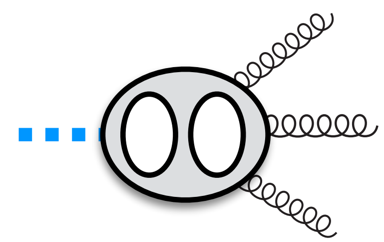
- Operators also appear as interaction vertices in effective field theories (EFT)



$$m_t \to \infty$$

Effective gluon-Higgs vertex:

$$\mathcal{L}_{\text{eff}} = C_0 H \text{tr}(G^2) + \mathcal{O}\left(\frac{1}{m_{\text{t}}^2}\right)$$

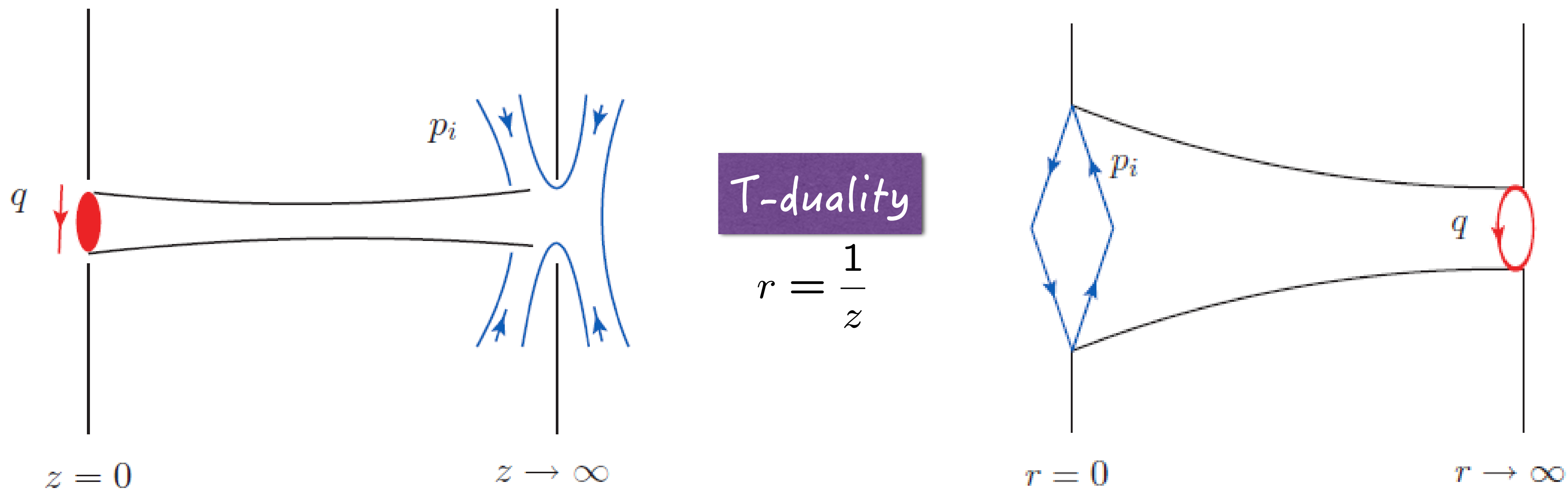
Higgs + multi-gluon scattering is a **form factors!** 

- Other connections and applications:

  Anomalous dimension, mass spectrum, critical exponents, integrability, …

Form factors at strong coupling

N=4 SYM $\xleftrightarrow{\text{AdS/CFT}}$ Type IIB string theory in $AdS_5 \times S^5$



Form factors as string minimal surfaces

Alday, Maldacena 2007

Y-system formulation

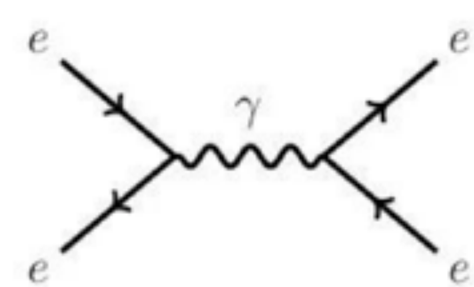
Indicate hidden structure

Maldacena, Zhiboedov 2010 (for AdS3)
Gao, GY 2013 (for AdS5)

Feynman diagram

Standard textbook method:

- universal
- simple rules
- intuitive picture



Feynman diagram

“Like the silicon chips of more recent years, the Feynman diagram was bringing computation to the masses.”

— Schwinger

Feynman diagram

"Like the silicon chips of more recent years, the Feynman diagram was bringing computation to the masses. Yes, one can analyze experience into individual pieces of topology. But eventually one has to put it all together again. And then the piecemeal approach loses some of its attraction."

— Schwinger

Feynman diagram

Practical application can be very complicated.

n-gluon tree amplitudes:

n	4	5	6	7	8	9	10
# graphs	4	25	220	2485	34300	559405	10525900

Surprising simplicity

Practical application can be very complicated.

n-gluon tree amplitudes:

n	4	5	6	7	8	9	10
# graphs	4	25	220	2485	34300	559405	10525900

n-gluon MHV tree amplitudes:

[Parke, Taylor, 1986]

$$\boxed{A_n^{\text{tree}}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+) = \frac{\langle ij\rangle^4}{\langle 12\rangle\cdots\langle n1\rangle}}$$

Written in spinor helicity formalism (Chinese Magic)
by Xu, Zhang, Chang 1984

Modern amplitudes methods

"A Renaissance of the S-Matrix Program"

S-matrix program

Wheeler 1937
Heisenberg 1943

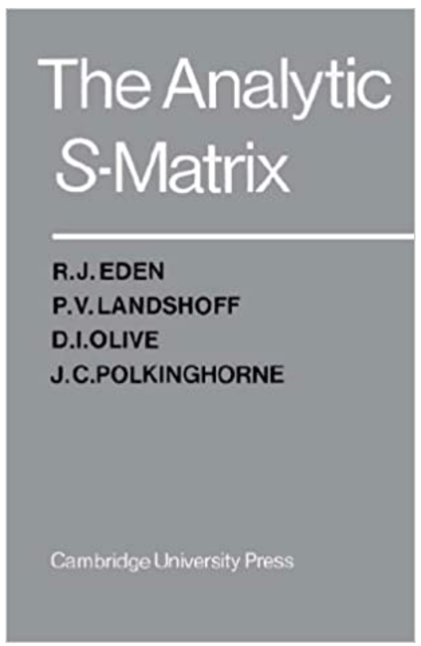
→

S-matrix bootstrap by
Chew, Mandelstam, etc
1950s-1960s

→

Modern amplitudes
On-shell methods

S-matrix program



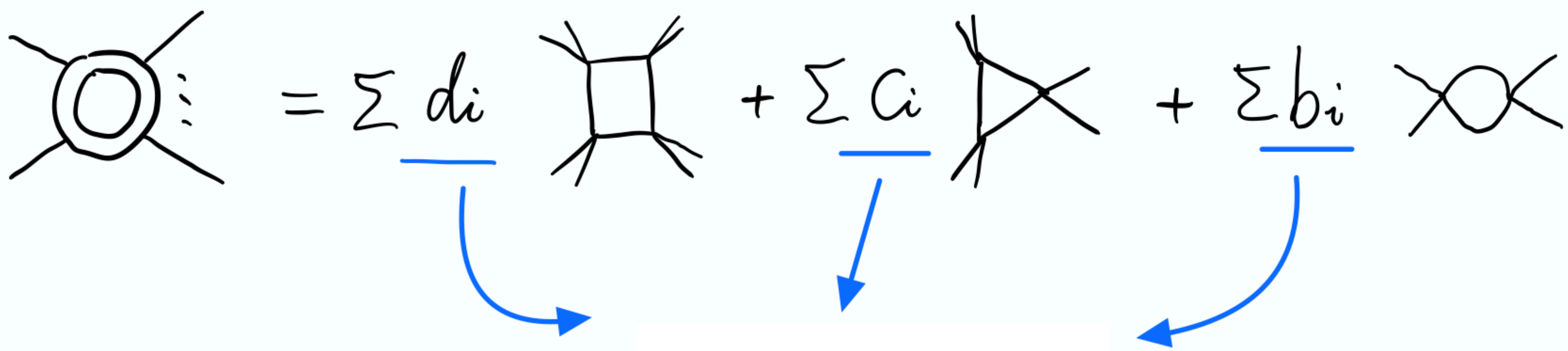
"The S-matrix is a Lorentz-invariant analytic function of all momentum variables with only those singularities required by unitarity."

"One should try to calculate S-matrix elements directly, without the use of field quantities, by requiring them to have some general properties that ought to be valid, …."

— Eden et.al, "The Analytic S-matrix", 1966

One-loop structure

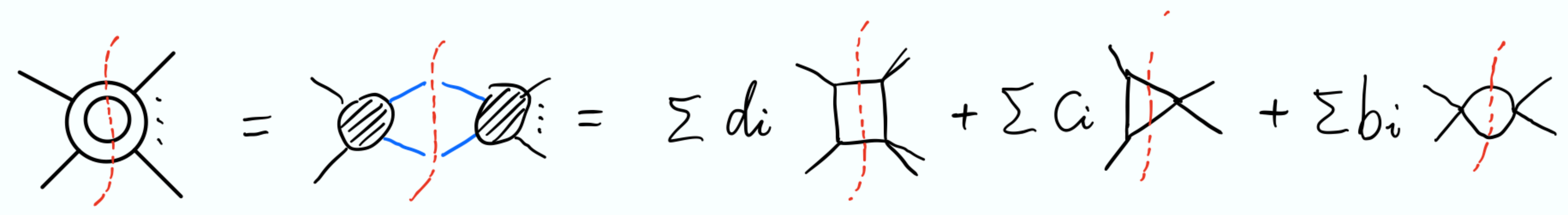
Consider one-loop amplitudes:



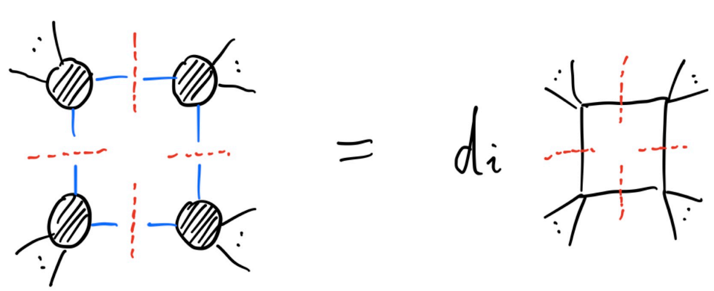
What we really want

Unitarity cuts

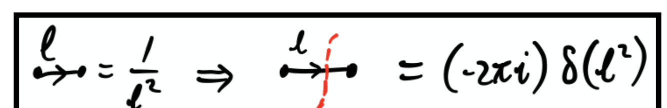
Using simpler tree-level blocks, one can derive the coefficients more efficiently:



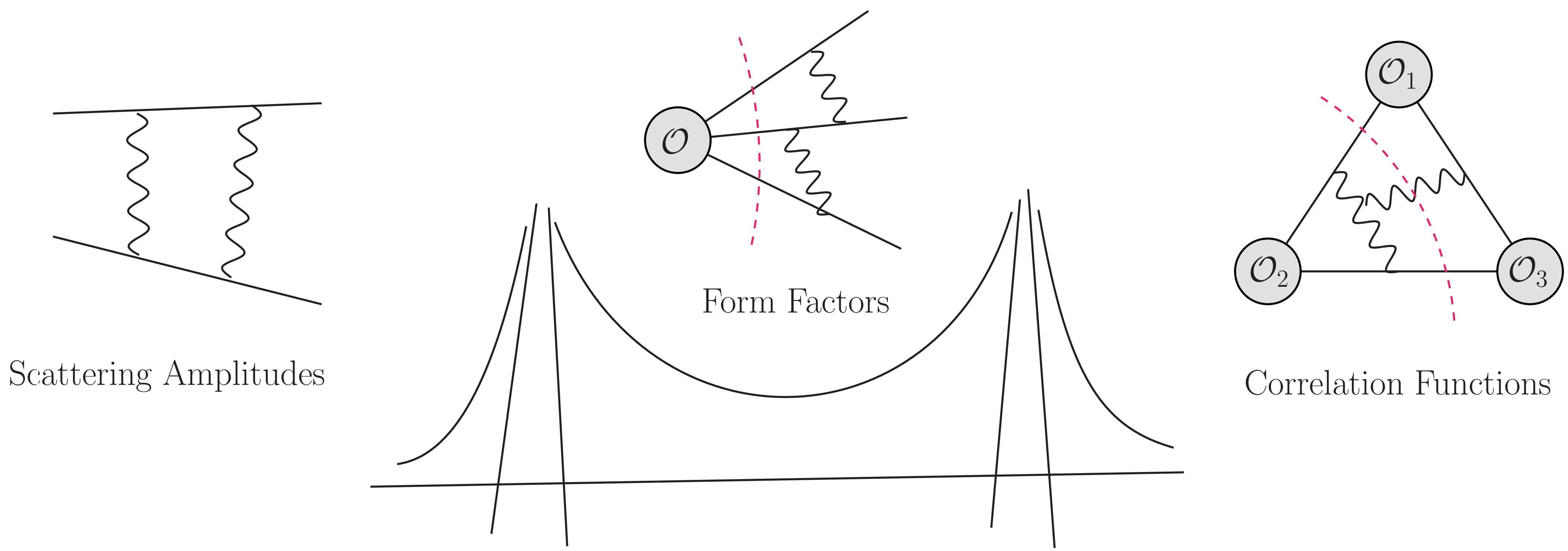
[Bern, Dixon, Dunbar, Kosower 1994]



generalized multiple cuts [Britto, Cachazo, Feng 2004]

Cutkosky cutting rule: 

On-shell methods



On-shell methods can be applied to operators and study EFT, for both the operator construction and high-loop renormalization.

Outline

- Introduction to form factor
- Master-bootstrap and MTP
- CK-duality and double-copy

- 2106.01374 [PRL (2021)], Yuanhong Guo, Lei Wang, GY
- 2205.12969, Yuanhong Guo, Qingjun Jin, Lei Wang, GY

Master bootstrap method

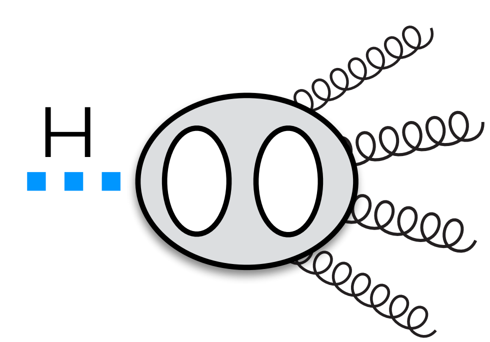
Ansatz in master integral expansion → **Physical constraints** → **Solution of coefficients**

$$\mathcal{F}^{(l),\mathrm{ansatz}} = \sum_i C_i I_i^{(l)}$$

- IR divergences
- Collinear factorization
- Spurious-pole cancellation
- Unitarity cut

$$\mathcal{F}^{(l),\mathrm{ansatz}} = \sum_i {\color{red}C_i} I_i^{(l)}$$

We apply this strategy to a frontier two-loop five-point scattering problem (Higgs plus four partons in N=4 SYM):



Outline of two-loop computation

Ansatz in master integral expansion

$$\mathcal{F}^{(2),\text{ansatz}}_{\text{tr}(\phi_{12}^3),4} = \sum_i C_i I_i^{(l)}$$

Guo, Wang, GY PRL 2021

Constraints	Parameters left
Symmetry of $(p_1 \leftrightarrow p_3)$	221
IR (Symbol)	82
Collinear limit (Symbol)	38
Spurious pole (Symbol)	22
IR (Function)	17
Collinear limit (Funcion)	10
If keeping only to ϵ^0 order	6
Simple unitarity cuts	0

Solution of coefficients

$$\mathcal{F}^{(2)}_{\text{tr}(\phi_{12}^3),4} = \sum_i C_i I_i^{(l)}$$

Master bootstrap method

Ansatz in master integral expansion → **Physical constraints** → Solution of coefficients

$$\mathcal{F}^{(l),\mathrm{ansatz}} = \sum_i C_i \, I_i^{(l)}$$

- IR divergences
- Collinear factorization
- Spurious-pole cancellation
- Unitarity cut

$$\mathcal{F}^{(l),\mathrm{ansatz}} = \sum_i {\color{red}C_i} \, I_i^{(l)}$$

The strategy does not rely on special symmetries of the theory, thus can be applied to general theories.

Maximal Transcendentality Principle

Maximal transcendentality principle

N=4 SYM $\longleftrightarrow$ **QCD**

Kotikov, Lipatov, 2001

For certain physical quantities, the maximally transcendental parts are equal between the two theories:

- Anomalous dimensions Kotikov, Lipatov, Onishchenko, Velizhanin 2004
- Form factors Brandhuber, Travaglini, GY 2012
- Wilson lines Li, Manteuffel, Schabinger, Zhu 2014

transcendental degree k

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k} = \int_0^z \frac{\mathrm{Li}_{k-1}(t)}{t} dt \qquad \zeta_k = \sum_{n=1}^{\infty} \frac{1}{n^k}$$

Two-loop Higgs to 3-gluon

$$
\begin{array}{l}
-2G(0,0,1,0,u)+G(0,0,1-v,1-v,u)+2G(0,0,-v,1-v,u)-G(0,1,0,1-v,u)+4G(0,1,1,0,u)-G(0,1,1-v,0,u)+G(0,1-v,0,1-v,u)\\
+G(0,1-v,1-v,0,u)-G(0,1-v,-v,1-v,u)+2G(0,-v,0,1-v,u)+2G(0,-v,1-v,0,u)-2G(0,-v,1-v,1-v,u)-2G(1,0,0,1-v,u)\\
-2G(1,0,1-v,0,u)+4G(1,1,0,0,u)-4G(1,1,1,0,u)-2G(1,1-v,0,0,u)+G(1-v,0,0,1-v,u)-G(1-v,0,1,0,u)-2G(-v,1-v,1-v,u)H(0,v)\\
-2G(1-v,1,0,0,u)+2G(1-v,1,0,1-v,u)+2G(1-v,1,1-v,0,u)+G(1-v,1-v,0,0,u)+2G(1-v,1-v,1,0,u)-2G(1-v,1-v,-v,1-v,u)\\
-G(1-v,-v,1-v,0,u)+4G(1-v,-v,-v,1-v,u)-2G(-v,0,1-v,1-v,u)-2G(-v,1-v,0,1-v,u)-2G(-v,1-v,1-v,0,u)+4G(1,0,1,0,u)\\
+4G(-v,-v,1-v,1-v,u)-4G(-v,-v,-v,1-v,u)-G(0,0,1-v,u)H(0,v)-G(0,1,0,u)H(0,v)-G(0,1-v,0,u)H(0,v)+G(0,1-v,1-v,u)H(0,v)\\
-G(0,-v,1-v,u)H(0,v)-2G(1,0,0,u)H(0,v)+G(1,0,1-v,u)H(0,v)+G(1,1-v,0,u)H(0,v)+G(1-v,0,0,u)H(0,v)-G(1-v,0,1-v,u)H(0,v)\\
-G(1-v,1,0,u)H(0,v)-G(1-v,1-v,0,u)H(0,v)-G(1-v,-v,1-v,u)H(0,v)+G(-v,0,1-v,u)H(0,v)+G(-v,1-v,0,u)H(0,v)+H(1,0,0,1,v)\\
-G(0,0,1-v,u)H(1,v)-G(0,0,-v,u)H(1,v)+G(0,1,0,u)H(1,v)-G(0,1-v,0,u)H(1,v)+G(0,1-v,-v,u)H(1,v)-2G(0,-v,0,u)H(1,v)\\
+2G(0,-v,1-v,u)H(1,v)\cdots u)H(1,v)-2G(1-v,1,0,u)H(1,v)-G(1-v,0,-v,1-v,u)\\
+G(1-v,-v,0,u)H(1,v)\cdots,1-v,0,u)H(1,v)-4G(-v,1-v,-v,u)H(1,v)\\
-4G(-v,-v,1-v,u)H(1,\cdots H(0,0,v)+G(1-v,0,u)H(0,0,v)+H(1,0,1,0,v)\\
-G(0,0,u)H(0,1,v)+G(0\cdots-2G(1-v,1-v,u)H(0,1,v)-3G(1-v,-v,u)H(0,1,v)\\
-G(-v,0,u)H(0,1,v)-2G\cdots 0,v)+G(0,-v,u)H(1,0,v)-G(1,0,u)H(1,0,v)\\
+2G(1-v,0,u)H(1,0,v)\cdots u)H(1,0,v)+2G(-v,1-v,u)H(1,0,v)+G(0,0,u)H(1,1,v)\\
-2G(0,-v,u)H(1,1,v)-2G(-v,0,u)H(1,1,v)+4G(-v,-v,u)H(1,1,v)+G(0,u)H(0,0,1,v)-3G(1-v,u)H(0,0,1,v)+4G(-v,u)H(0,0,1,v)\\
+G(0,u)H(0,1,0,v)+G(1-v,u)H(0,1,0,v)-G(0,u)H(0,1,1,v)+2G(-v,u)H(0,1,1,v)+G(0,u)H(1,0,0,v)+G(1-v,u)H(1,0,0,v)+H(1,1,0,0,v)\\
-G(0,u)H(1,0,1,v)+2G(-v,u)H(1,0,1,v)-G(0,u)H(1,1,0,v)+4G(1-v,u)H(1,1,0,v)-2G(-v,u)H(1,1,0,v)+H(0,0,1,1,v)+H(0,1,0,1,v)\\
+G(1-v,1-v,u)H(0,0,v)+2G(1-v,1-v,-v,u)H(1,v)-G(1-v,-v,0,1-v,u)+H(0,1,1,0,v)+G(1-v,0,1-v,0,u)-G(0,1-v,1,0,u)\\
+4G(-v,1-v,-v,1-v,u)
\end{array}
$$

Maximal transcendental part of QCD

[Gehrmann, Jaquier, Glover, Koukoutsakis 2011]

$$\updownarrow$$

$$
\begin{aligned}
&-2\left[\mathrm{J}_4\left(-\frac{uv}{w}\right)+\mathrm{J}_4\left(-\frac{vw}{u}\right)+\mathrm{J}_4\left(-\frac{wu}{v}\right)\right]-8\sum_{i=1}^{3}\left[\mathrm{Li}_4\left(1-u_i^{-1}\right)+\frac{\log^4 u_i}{4!}\right]\\
&-2\left[\sum_{i=1}^{3}\mathrm{Li}_2(1-u_i)+\frac{\log^2 u_i}{2!}\right]^2+\frac{1}{2}\left[\sum_{i=1}^{3}\log^2 u_i\right]^2-\frac{\log^4(uvw)}{4!}-\frac{23}{2}\zeta_4
\end{aligned}
$$

[Brandhuber, Travaglini, GY 2012]

N=4 SYM

Master-bootstrap

Using the bootstrap strategy, we are able to prove all the previously observed maximally transcendental correspondence for Higgs amplitudes (i.e. form factors) and also find new examples.

$$\boxed{\begin{array}{c}\text{universal}\\ \text{constraints}\end{array}} \xrightarrow{\text{Bootstrap}} \boxed{\begin{array}{c}\text{universal}\\ \text{results}\end{array}}$$

If the "theory-independent" constraints are strong enough to fix the results, then the results must be also theory independent.

$$\mathcal{F}^{(L),\mathcal{N}=4}_{\mathcal{L}\sim\mathrm{tr}(F^2)}(1,2,3) = \mathcal{F}^{(L),\mathrm{QCD}}_{\mathrm{tr}(F^2),\mathrm{M.T.}}(1^g,2^g,3^g) = \mathcal{F}^{(L),\mathrm{QCD}}_{\mathrm{tr}(F^2),\mathrm{M.T.}}(1^q,2^{\bar{q}},3^g)\Big|_{C_F\to C_A}$$

$$\mathscr{F}^{(2),\mathcal{N}=4}_{\mathrm{tr}(F^3)}(1^-,2^-,3^-,4^+) = \mathscr{F}^{(2),\mathrm{QCD}}_{\mathrm{tr}(F^3)}(1^-,2^-,3^-,4^+)\Big|_{n_f\to 4N_c}$$ (New 4-point example)

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- 2106.01374 [PRL (2021)], 2111.03021, 2112.09123, Guanda Lin, Siyuan Zhang, GY
- 2111.12719, Guanda Lin, GY

Color-kinematics duality

In 2008 Bern, Carrasco and Johansson proposed an intriguing duality between color and kinematics factors:

Duality

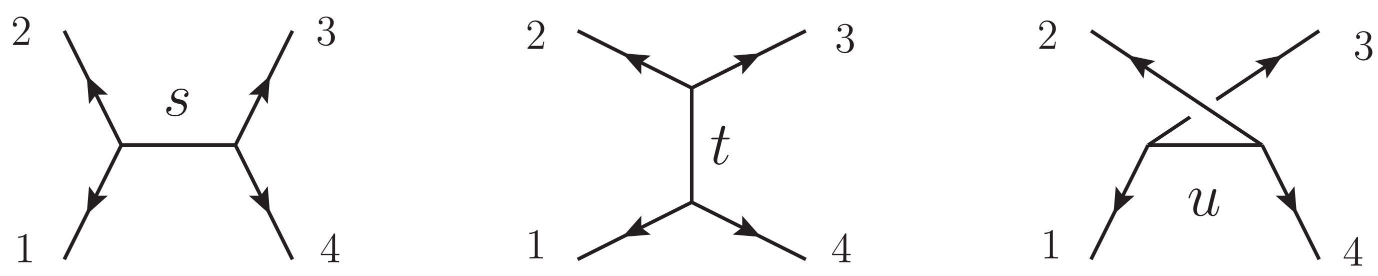
Color factor $\longleftrightarrow$ Kinematic factor

(conjecture)

$$\tilde{f}^{abc} = \mathrm{Tr}([T^a, T^b]T^c) \qquad\qquad s_{ij} = (p_i + p_j)^2$$

Gauge symmetry **Spacetime symmetry**

Example: 4-pt amplitude



$$A_4(1,2,3,4) = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}$$

$$c_s = \tilde{f}^{a_1 a_2 b} \tilde{f}^{b a_3 a_4}, \quad c_t = \tilde{f}^{a_2 a_3 b} \tilde{f}^{b a_4 a_1}, \quad c_u = \tilde{f}^{a_1 a_3 b} \tilde{f}^{b a_2 a_4}$$

$$c_s = c_t + c_u \quad \Rightarrow \quad n_s = n_t + n_u$$

Jacobi identity

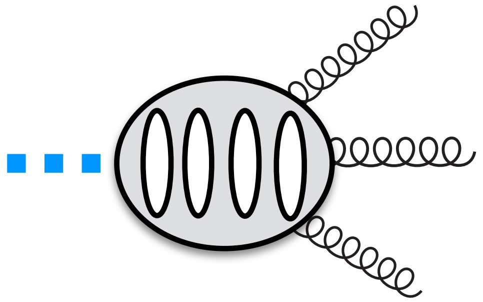
dual Jacobi relation

Three-point form factor

Physical quantity:

three-point form factor of stress-tensor multiplet in N=4 SYM:

$$\mathcal{F}_3 = \int d^4x\, e^{-iq\cdot x} \langle p_1, p_2, p_3 \,|\, \mathrm{tr}(F^2)(x) \,|\, 0 \rangle$$



CK duality

+

on-shell unitarity

$\longrightarrow$

full-color results

up to four loops

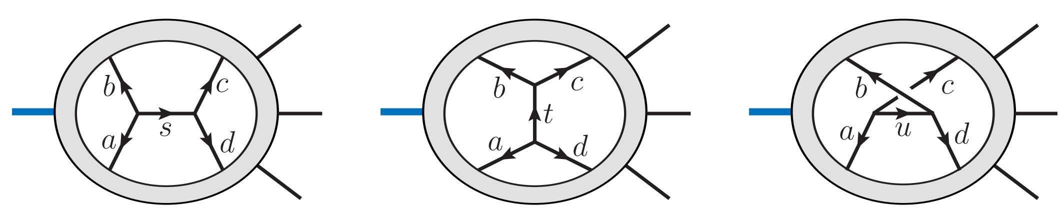
Such a computation would be very difficult using traditional Feynman diagram method.

Strategy of loop computation

CK-duality

$$\mathscr{F}^{(\ell)} \sim \sum_i \int \frac{C_i \times N_i}{\prod D}$$

Compact ansatz of the loop integrand



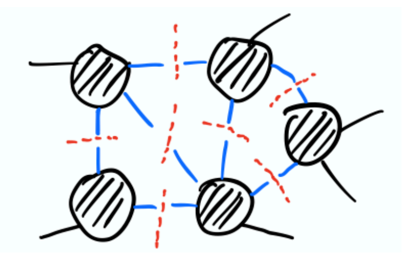
$$C_s = C_t + C_u \longrightarrow N_s = N_t + N_u$$

Strategy of loop computation

CK-duality

Conjecture

Ansatz of the loop integrand



Unitarity cuts

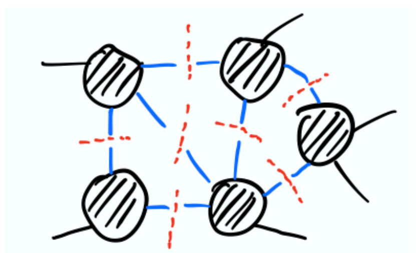
Solving linear equations

Strategy of loop computation

CK-duality

Conjecture

Ansatz of the loop integrand

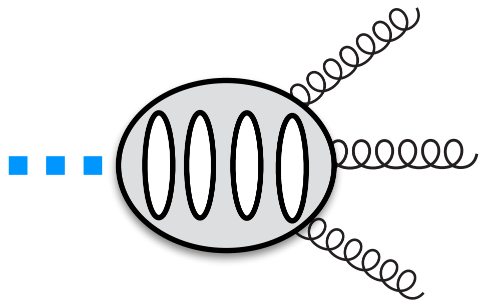


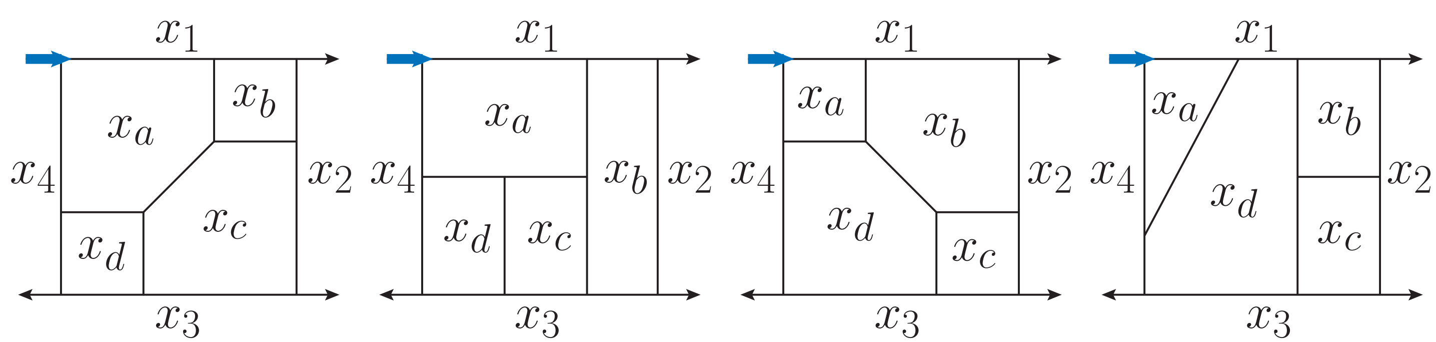
Unitarity cuts

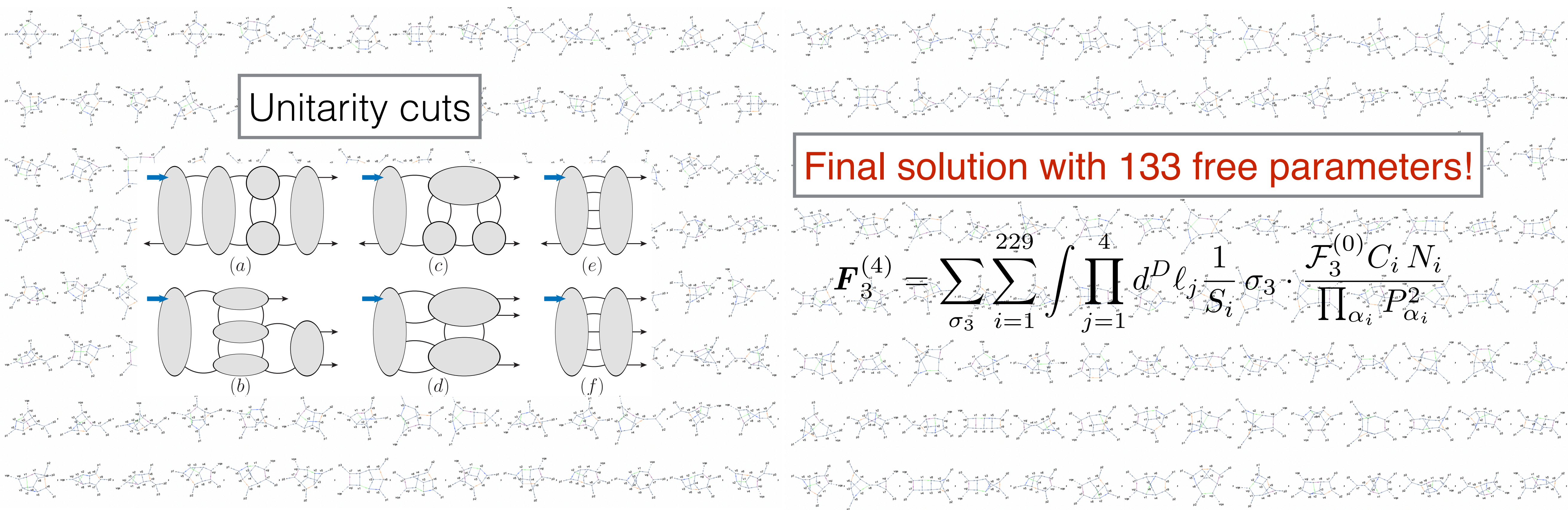
Solving linear equations

Main challenge: it is a prior not known whether the solution exists

Four loops





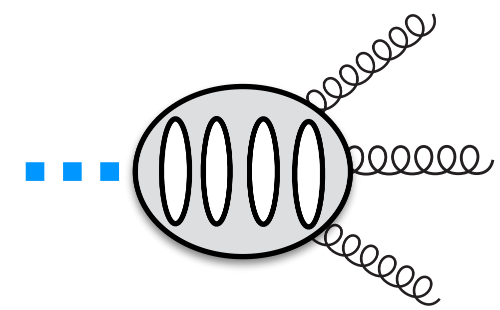


$$\boldsymbol{F}_3^{(4)} = \sum_{\sigma_3} \sum_{i=1}^{229} \int \prod_{j=1}^{4} d^D \ell_j \frac{1}{S_i} \sigma_3 \cdot \frac{\mathcal{F}_3^{(0)} C_i N_i}{\prod_{\alpha_i} P_{\alpha_i}^2}$$

G. Lin, GY, S. Zhang, 2112.09123

Results up to four loops

$$\mathscr{F}_3 = \int d^4x\, e^{-iq\cdot x} \langle p_1, p_2, p_3 \,|\, \mathrm{tr}(F^2)(x) \,|\, 0 \rangle$$



L loops	L=1	L=2	L=3	L=4
# of cubic graphs	2	6	29	229
# of planar masters	1	2	2	4
# of free parameters	1	4	24	133

It is promising to go to higher loops.

Lin, GY, Zhang PRL 2021, 2112.09123

In the large-N limit, the remainder function was computed recently to **8 loops** via symbol bootstrap and the (non-perturbative) OPE input.

Dixon, Gurdogan, McLeod, Wilhelm 2204.11901
Sever, Tumanov, Wilhelm, 2009.11297 (FFOPE)

Double-copy of form factor?

The double-copy of a local operator is not obvious: a “local” operator would break the diffeomorphism invariance in gravity.

$$\mathcal{O}(x) \quad \overset{?}{\to} \quad \int d^4x\,\mathcal{O}(x)$$

The solution is to impose CK duality.

$$\sum_a \frac{c_a\,(n_a|_{\varepsilon_i\to p_i})}{D_a} = 0 \qquad \longrightarrow \qquad \sum_a \frac{n_a\,(n_a|_{\varepsilon_i\to p_i})}{D_a} = 0$$

$$c_a = c_b + c_c \qquad\qquad n_a = n_b + n_c$$

An intriguing feature

The CK-dual numerators contain spurious poles for the gauge theory form factors. After double-copy, the spurious poles in gauge theory can become real physical poles in gravity.

Spurious poles $\xrightarrow{\text{Double-copy}}$ Real propagators

Gauge theory | Gravity theory

G. Lin and GY, 2111.12719

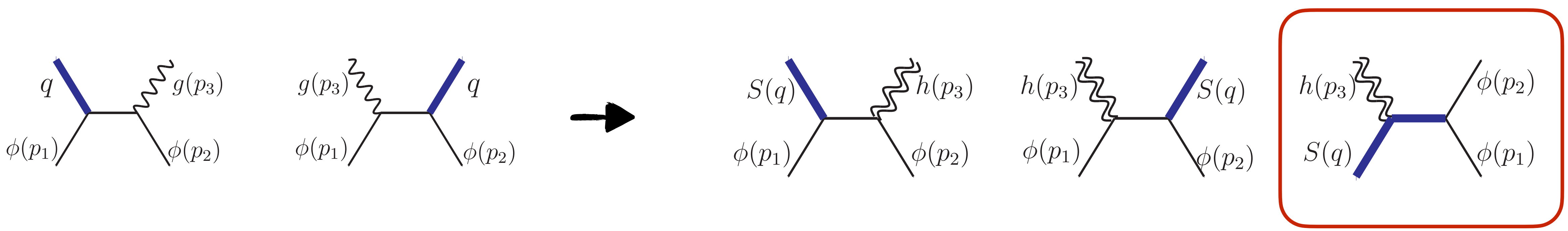
Example: 3-point form factor

$$\mathcal{G}_3 = \frac{(N_1^{\rm CK})^2}{s_{23}} + \frac{(N_2^{\rm CK})^2}{s_{13}} = \frac{s_{13}s_{23}}{s_{13}+s_{23}}\Big(\mathcal{F}_3(1^\phi, 3^g, 2^\phi)\Big)^2$$

There is a nice factorization behavior at the new pole:

$$s_{13} + s_{23} = q^2 - s_{12} = 0$$

$$\text{Res}\,[\mathcal{G}_3]_{s_{12}=q^2} = (\epsilon_3 \cdot q)^2 = \Big(\mathcal{F}_2(1^\phi, 2^\phi)\Big)^2 \times \big(\mathcal{A}_3(\mathbf{q}_2^S, 3^g, -q^S)\big)^2$$



A new graph in gravity

Hidden relation for gauge-theory form factors

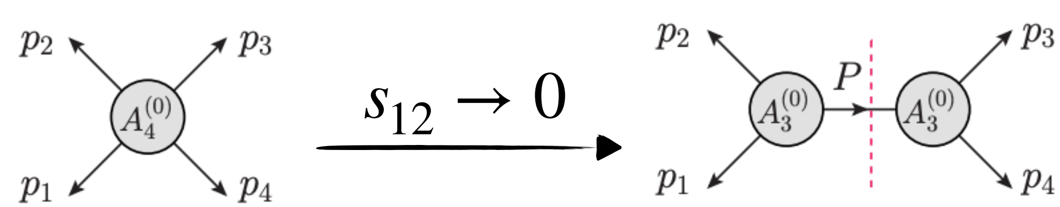
"Factorization" at spurious poles:

$$\vec{v}\cdot\vec{\mathcal{F}}_n\Big|_{\text{spurious pole}} = \mathcal{F}_m \times \mathcal{A}_{n+2-m}$$

$$\big[s_{42}\mathcal{F}_4(1,3,4,2) + (s_{42}+s_{43})\mathcal{F}_4(1,4,3,2)\big]\big|_{s_{123}=q^2} = \mathcal{F}_3(1^\phi,3^g,2^\phi)\ \mathcal{A}_3(\mathbf{q}_3^S,4^g,-q^S)$$

Not a residue!

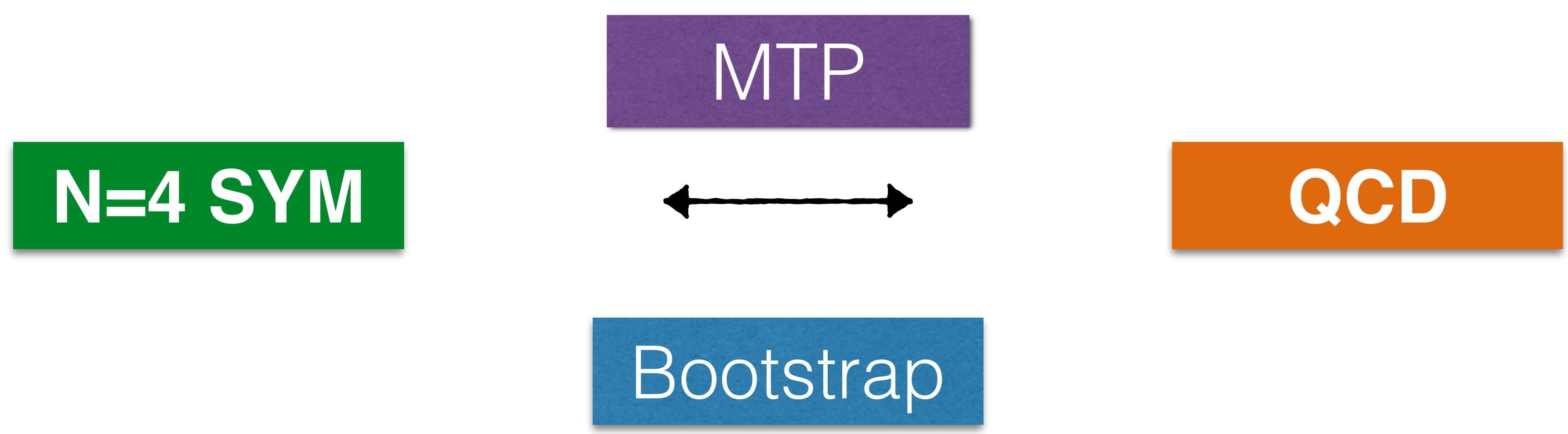
Comparing to the usual factorization:



$$\mathrm{Res}_{s_{12}\to 0}A_4 = A_3 \times A_3$$

Summary and outlook

Summary and outlook



- MTP for other observables, e.g. anomalous dimensions?
- Lower transcendental part?

Summary and outlook

Spurious poles $\xrightarrow{\text{Double-copy}}$ Real propagators

Gauge theory | Gravity theory

$$\vec{v} \cdot \vec{\mathcal{F}}_n \Big|_{\text{spurious pole}} = \mathcal{F}_m \times \mathcal{A}_{n+2-m}$$

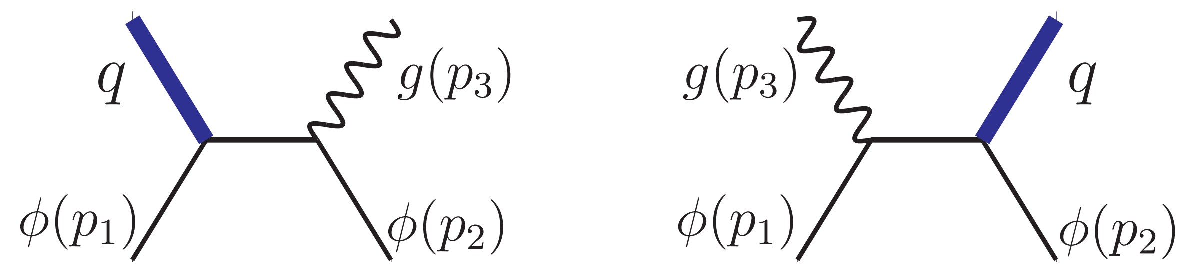
- How about loop level?

Thank you for your attention!

Extra slides

Example: 3-point form factor

$$F_3 = \int d^4x\, e^{-iq\cdot x} \langle p_1^{\phi}, p_2^{\phi}, p_3^{g} \,|\, \mathrm{tr}(\phi^2)(x) \,|\, 0 \rangle$$



$$\boldsymbol{F}_3(1^{\phi}, 2^{\phi}, 3^{g}) = \frac{C_1 N_1}{s_{23}} + \frac{C_2 N_2}{s_{13}}$$

$$C_1 = C_2 = f^{a_1 a_2 a_3} \longrightarrow N_1^{\mathrm{CK}} = N_2^{\mathrm{CK}} = \frac{s_{13}s_{23}}{s_{13}+s_{23}} \mathcal{F}_3(1^{\phi}, 3^{g}, 2^{\phi})$$

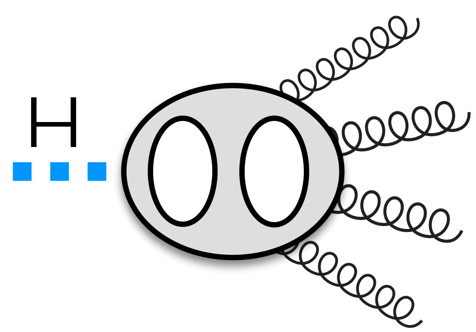
Unique solution with a spurious pole

$$\mathcal{G}_3 = \frac{(N_1^{\mathrm{CK}})^2}{s_{23}} + \frac{(N_2^{\mathrm{CK}})^2}{s_{13}} = \frac{s_{13}s_{23}}{s_{13}+s_{23}} \Big(\mathcal{F}_3(1^{\phi}, 3^{g}, 2^{\phi}) \Big)^2$$

Manifestly diffeomorphism invariant

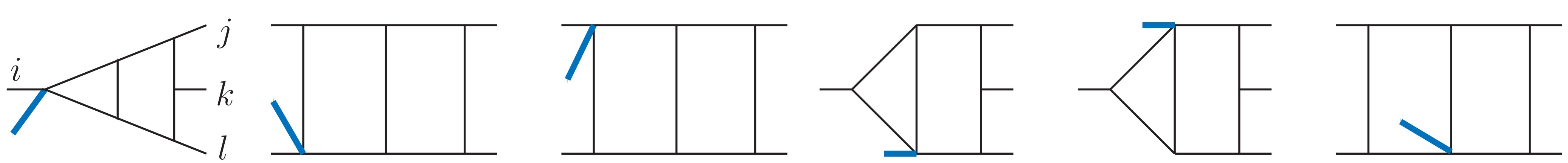
Ansatz of the form factors

Our result provides a first two-loop five-point example with a color-singlet off-shell leg.



$$\mathcal{F}_{\mathcal{O},4} = \int d^4x\, e^{-iq\cdot x}\langle 1,2,3,4\,|\,\mathcal{O}(x)\,|\,0\rangle$$

$$\{s_{12}, s_{23}, s_{34}, s_{14}, s_{13}, s_{24}, \mathrm{tr}_5\}; \qquad \mathrm{tr}_5 = 4i\varepsilon_{p_1p_2p_3p_4}$$



Physical constraints

IR divergences

$$\log \mathcal{I}^{\text{ren}} = -\sum_{l=1}^{\infty} g^{2l} \left[\frac{\gamma_{\text{cusp}}^{(l)}}{(2l\epsilon)^2} + \frac{\mathcal{G}_{\text{coll}}^{(l)}}{2l\epsilon} \right] \sum_{i=1}^{n} (-s_{ii+1})^{-l\epsilon} \cdot \mathbb{1} + \mathcal{O}(\epsilon^0)$$

Collinear factorization

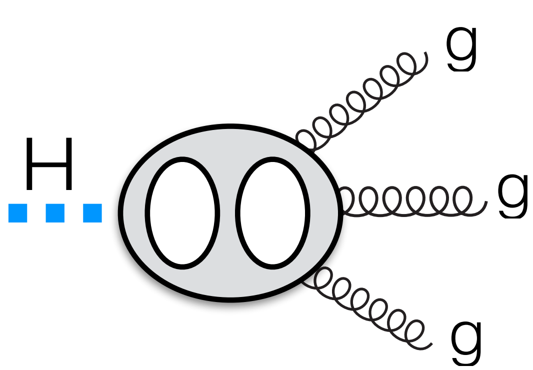
$$\mathcal{F}_n^{(L)}(1,\ldots,a^{h_a},b^{h_b},\ldots,n) \xrightarrow{p_a||p_b} \sum_{\ell}\sum_{\sigma} \mathbf{Sp}_{-\sigma}^{(\ell)}(a^{h_a},b^{h_b})\, \mathcal{F}_{n-1}^{(L-\ell)}(1,\ldots,(a+b)^{\sigma},\ldots,n)$$

$$\mathbf{Sp}^{(1)}(P\to a\,b;z) = \mathbf{Sp}^{(0)}(P\to a\,b;z)\; r_1^{[1],\text{MT}}(P^2,z) + (\text{lower transendental part})$$

$$r_1^{[1],\text{MT}}(P^2,z) = \frac{e^{\epsilon\gamma_{\text{E}}}\Gamma(-\epsilon)^2\Gamma(\epsilon+1)}{\Gamma(1-2\epsilon)}(-P^2)^{-\epsilon}\Big\{1 - z^{-\epsilon} - (1-z)^{-\epsilon} + \epsilon^2\big[\log(z)\log(1-z) - \zeta_2\big] + \mathcal{O}(\epsilon^3)\Big\}$$

The maximally transcendental parts of IR divergences and collinear splitting factors are universal for general gauge theories.

Three-point next-to-mini form factor



External particles	$(1^-, 2^-, 3^-)$	$(1^-, 2^-, 3^+)$	$(1^q, 2^{\bar{q}}, 3^-)$
Constraints	Remaining parameters		
Starting ansatz	89	89	89
Symmetry	24	53	89
IR	11	21	48
Collinear limit	1	5	21
Color factor	0	2	21
Smooth light-like limit of q	0	0	11

IR & collinear constraints are enough to fix 3-gluon results.

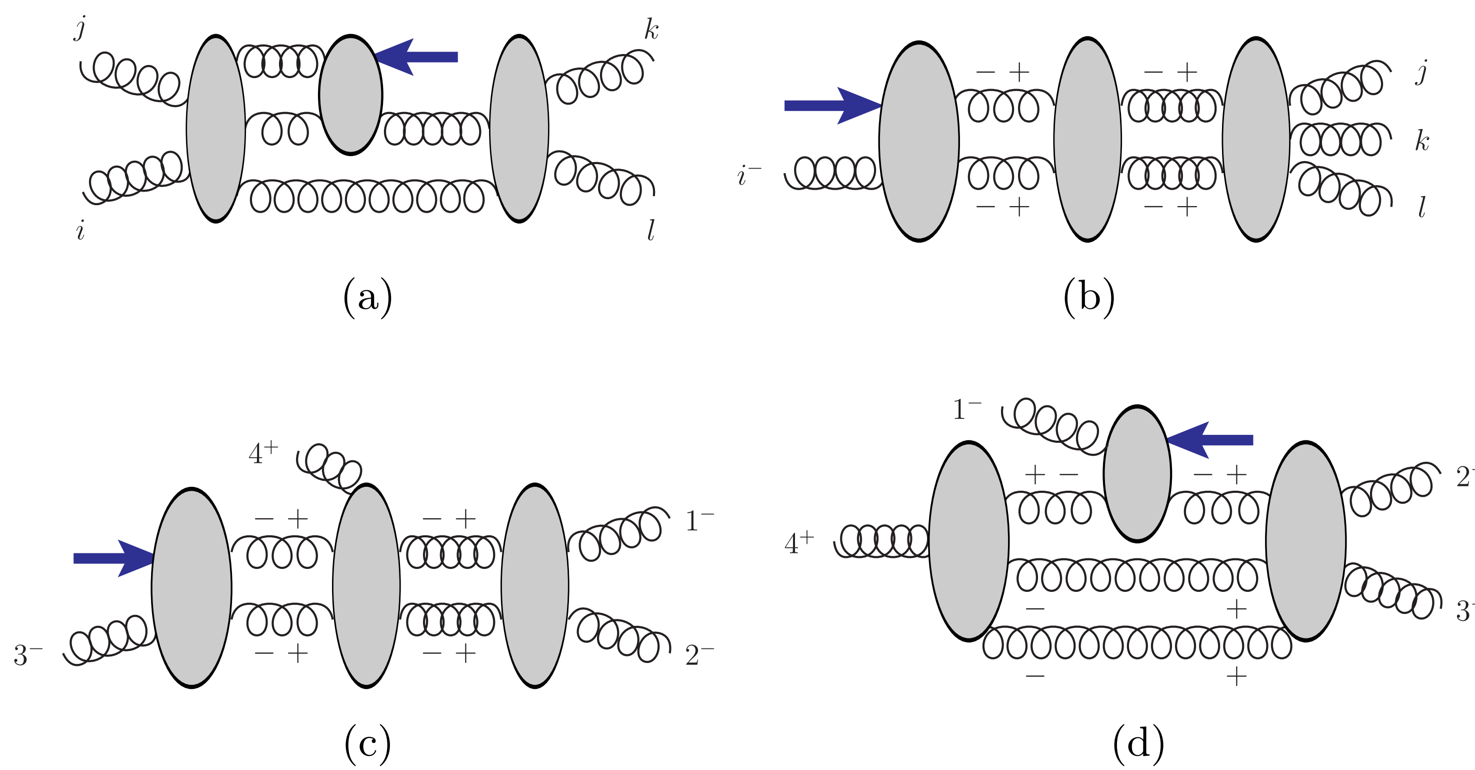
$$\mathcal{F}^{(L),\mathcal{N}=4}_{\mathcal{L}\sim\mathrm{tr}(F^2)}(1,2,3) = \mathcal{F}^{(L),\mathrm{QCD}}_{\mathrm{tr}(F^2),\mathrm{M.T.}}(1^g,2^g,3^g) = \mathcal{F}^{(L),\mathrm{QCD}}_{\mathrm{tr}(F^2),\mathrm{M.T.}}(1^q,2^{\bar{q}},3^g)\Big|_{C_F\to C_A}$$

There is no fermion or scalar contribution, so the result is also the same as in the pure YM theory.

Physical constraints

There are universal cuts that involve only gluon states and thus are also universal for general gauge theories.

Unitarity cut



A counterexample of MTP

One-loop four-gluon amplitudes do not obey MTP:

$$A^{(1),\mathrm{M.T.}}_{4,\mathrm{YM}}(1^-,2^+,3^-,4^+)\,|_{\mathrm{IR}} = A^{(1)}_{4,\mathcal{N}=4}\,|_{\mathrm{IR}}$$

$$A^{(1),\mathrm{M.T.}}_{4,\mathrm{YM}}(1^-,2^+,3^-,4^+)\,|_{\mathrm{fin}} \neq A^{(1)}_{4,\mathcal{N}=4}\,|_{\mathrm{fin}}$$

$$A^{(1)}_{4,\mathcal{N}=4} = A^{(1)}_{4,\mathrm{gluon}} + A^{(1)}_{4,\mathrm{fermion}} + A^{(1)}_{4,\mathrm{scalar}}$$

Four-point next-to-mini form factor

$$\begin{aligned}\mathcal{F}_{\mathrm{tr}(F^3),4} :=& \mathcal{F}_{\mathrm{tr}(F^3),4}(1^-,2^-,3^-,4^+;q) \\ =& \int d^D x e^{-iq\cdot x} \langle g_-(p_1) g_-(p_2) g_-(p_3) g_+(p_4) | \mathrm{tr}(F^3)(x) | 0 \rangle\end{aligned}$$

Bootstrap for the next-to-minimal four-point form factor in pure YM

Two-loop case:

Constraints	Parameters left
Starting ansatz	1105
Symmetry of $(p_1 \leftrightarrow p_3)$	560
IR (Symbol)	207
Collinear limit (Symbol)	119
Spurious pole (Symbol)	53
IR (Function)	40
Collinear limit (Funcion)	24
Spurious pole (Funcion)	20
Simple unitarity cuts	0

MHV tree form factors

MHV structure of form factors:

Brandhuber, Spence, Travaglini, GY 2010

$$F_n^{\mathrm{MHV}}(1^+, .., i_\phi, .., j_\phi, .., n^+; \mathrm{tr}(\phi^2)) = \delta^4(\sum_{i=1}^{n} p_i - q) \frac{\langle ij \rangle^2}{\langle 12 \rangle \cdots \langle n1 \rangle}$$

$$\boxed{q = \sum_i p_i, \quad p_i^2 = 0, \quad q^2 \neq 0}$$

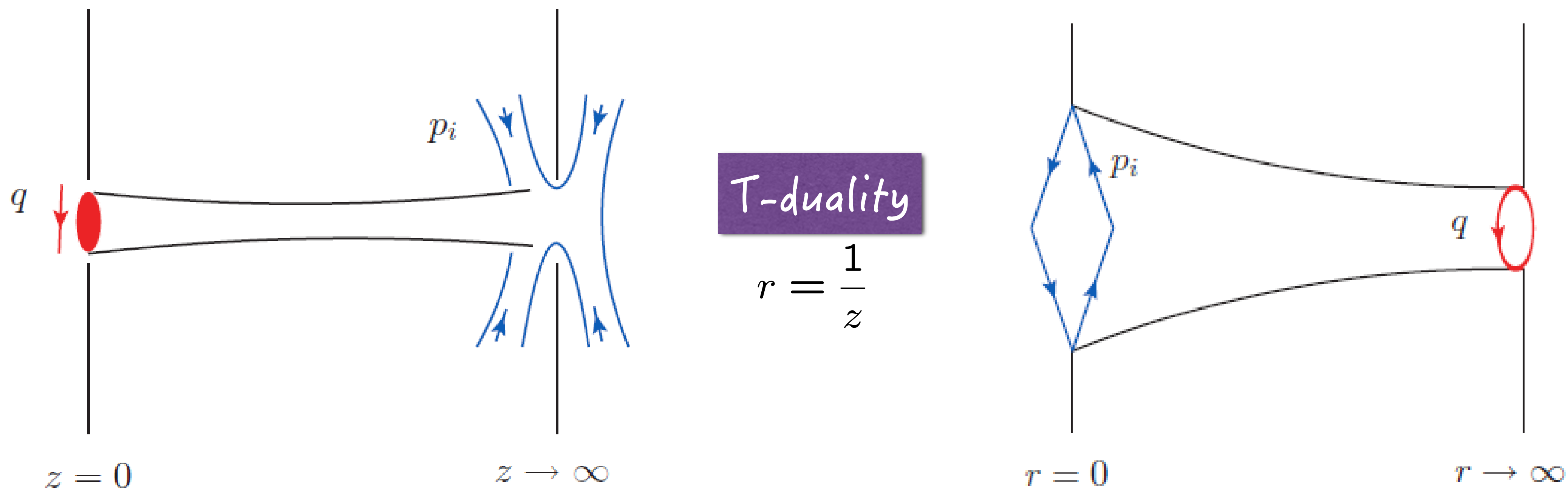
Compare with Parke-Taylor formula for amplitudes:

$$A_n^{\mathrm{MHV}}(1^+, .., i^-, .., j^-, .., n^+) = \delta^4(\sum_{i=1}^{n} p_i) \frac{\langle ij \rangle^4}{\langle 12 \rangle \cdots \langle n1 \rangle}$$

$$\boxed{0 = \sum_i p_i, \quad p_i^2 = 0}$$

Form factors at strong coupling

N=4 SYM $\xleftrightarrow{\text{AdS/CFT}}$ Type IIB string theory in $AdS_5 \times S^5$



Form factors as string minimal surfaces

Alday, Maldacena 2007

Y-system formulation

Indicate hidden structure

Maldacena, Zhiboedov 2010 (for AdS3)
Gao, GY 2013 (for AdS5)