Binary Dynamics from Worldline QFT for Scalar-QED

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Outline

- Introduction
- WQFT formalism for scalar-QED
- Generating function emerging from WQFT up to 3PM
- Scattering angle
- Conclusions and outlook

Introduction

- Gravitational-wave physics: dynamics of binary systems of black holes and neutron stars in classical gravity
  [LIGO/Virgo] [Pretorius, Campanelli, Lousto, Marronetti, Zlochower, Baker, Centrella, Koppitz, van Meter, ….]
  - Growing demands for high-precision **numerical and analytical** results

- Gravity approaches: EOB, NRGR, EFT-based approaches…
  [Buonanno, Damour, Goldberger, Rothstein, Kol, Smolkin, Gilmore, Ross, Foffa, Sturani, Mastrolia, Sturm, Porto, Blümlein, Maier, Marquard, Schäfer, …]

- "Amplitude-based" approaches (loosely defined):
  - EFT matching: extracting classical observables (e.g. momentum transfer, spin kick, etc) from quantum scattering amplitudes; computing generating functions from amplitudes (eikonal & radial action)
    [Bern, Cheung, Roiban, Shen, Solon, Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove, Neill, Rothstein, Zeng, Parra-Martinez, Ruf, …]
  - Observable-based KMOC formalism
    [Kosower, Maybee, O'Connell]
  - Heavy-mass EFT
    [Brandhuber, Chen, Travaglini, Wen]
  - **Wordline QFT formalism**
    [Mogull, Plefka, Steinhoff]
    - State-of-the-arts classical observables for spinning binary systems at 3PM [Jakobsen, Mogull, Plefka, Steinhoff]
    - Subtleties arise when it comes to generating functions

WQFT Formalism for Scalar-QED

- Scalar-QED as a toy model

- Worldline action for a charged massive non-spinning point-particle in an electromagnetic background
  [Schubert, Edwards]

$$\mathcal{S}_i = -m_i \int \mathrm{d}\tau \left[\frac{1}{2}(\eta^{-1}\dot{x}_i^2 + \eta) + ie\frac{q_i}{m_i} A_\mu \dot{x}_i^\mu \right]$$

Mass: m_i
Electric charge: q_i
Photon: A_μ

  - Worldline coordinate parameterised by the proper time $\dot{x}^\mu = \mathrm{d}x^\mu / \mathrm{d}\tau$
  - For convenience, we choose the einbein $\eta(\tau) = 1$

- Usual EM action in the bulk

- Specialise to the plane waves of fixed momenta and polarisations $A_\mu(x) = \sum_{i=1}^{n} \varepsilon_{i\mu} e^{ik_i \cdot x}$
  [Schubert, Edwards]
- Expand the wordline around the straight line trajectory $x_i^\mu = b_i^\mu + u_i^\mu \tau + z_i^\mu(\tau)$

- Photon-dressed Feynman-Schwinger propagator can be identified with the path integral for the WQFT correlation, with external legs amputated through the LSZ reduction [Mogull, Plefka, Steinhoff]

WQFT Feynman Rules in Frequency-Momentum Space

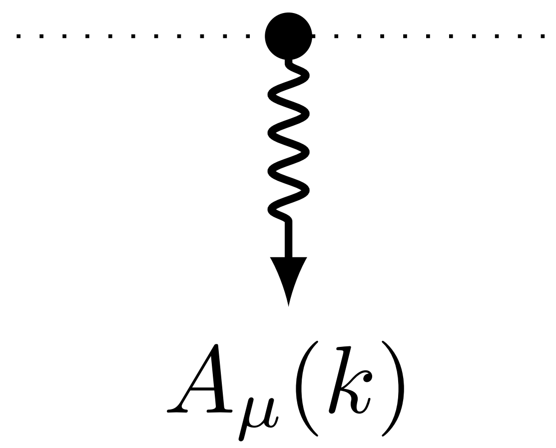
- Fourier transform to frequency-momentum space
  [Mogull, Plefka, Steinhoff]

$$z_i^\mu(\tau) = \int_\omega e^{-i\tau\omega} z_i^\mu(\omega) \qquad A_\mu(x) = \int_k e^{-ik\cdot x} A_\mu(-k)$$

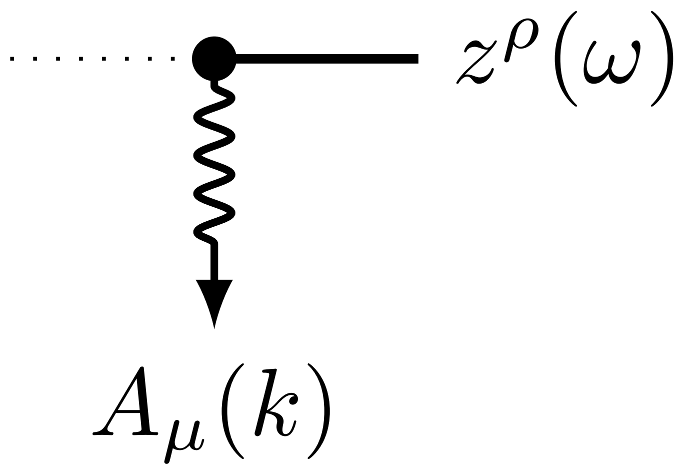
- Interaction term

$$\mathcal{S}_{\rm int} = \sum_{n=0}^{\infty} \frac{i^n e\, q_a}{n!} \int_{k,\omega_1,\cdots,\omega_n} e^{ik\cdot b}\hat{\delta}\left(k\cdot u + \sum_i \omega_i\right) \prod_i z^{\rho_i} A_\mu(-k) \left(\prod_i k_{\rho_i} u^\mu + \sum_i \omega_i \delta^\mu_{\rho_i} \prod_{j\neq i} k_{\rho_j}\right)$$

- Wordline-photon interaction vertices

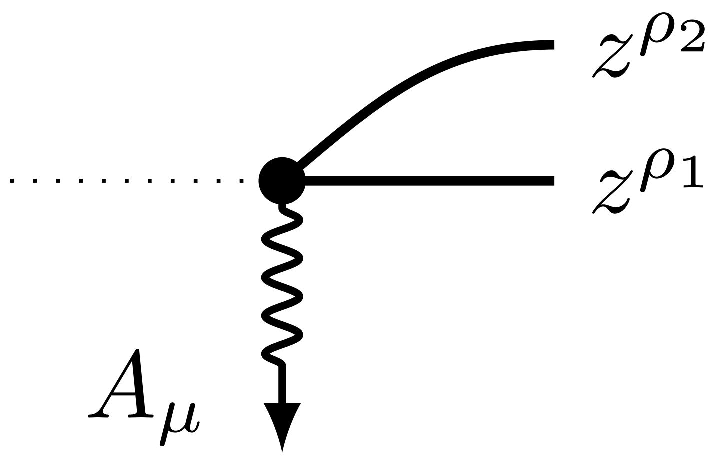


$$= -ie\, q_a \exp(ik\cdot b)\, \hat{\delta}(k\cdot u) u^\mu$$



$$= e\, q_a e^{ik\cdot b} \hat{\delta}(k\cdot u + \omega) \left(k_\rho u^\mu + \omega \delta^\mu_\rho\right)$$

WQFT Feynman Rules in Frequency-Momentum Space



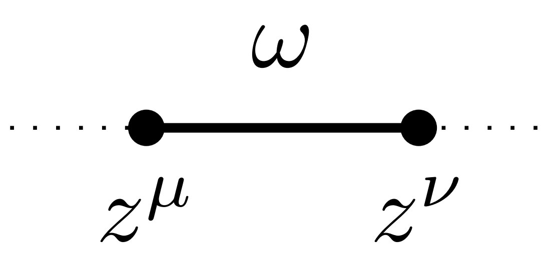
$$= ie\, q_a e^{ik\cdot b} \bar{\delta}\left(k\cdot u + \omega_1 + \omega_2\right)\left(k_{\rho_1}k_{\rho_2}u^\mu + \omega_1 \delta^\mu_{\rho_1} k_{\rho_2} + \omega_2 \delta^\mu_{\rho_2} k_{\rho_1}\right)$$

- Recursion relation of the worldline-photon vertices

$$\frac{\partial}{\partial b^\nu} V^\mu_{\rho_1\cdots\rho_n}(k, \omega_1, \cdots, \omega_n) = V^\mu_{\rho_1,\cdots,\rho_{n+1}}(k, \omega_1, \cdots, \omega_n, 0)\Big|_{\rho_{n+1}=\nu}$$

- Propagators
  - Photon in the bulk: usual Feynman propagator
  - Worldline propagator
    - Time-symmetric propagator
    - But ieps-prescription drops out in the end

[Brandhuber, Chen, Travaglini, Wen] [Shi, Plefka] [Bjerrum-Bohr, Planté, Vanhove]

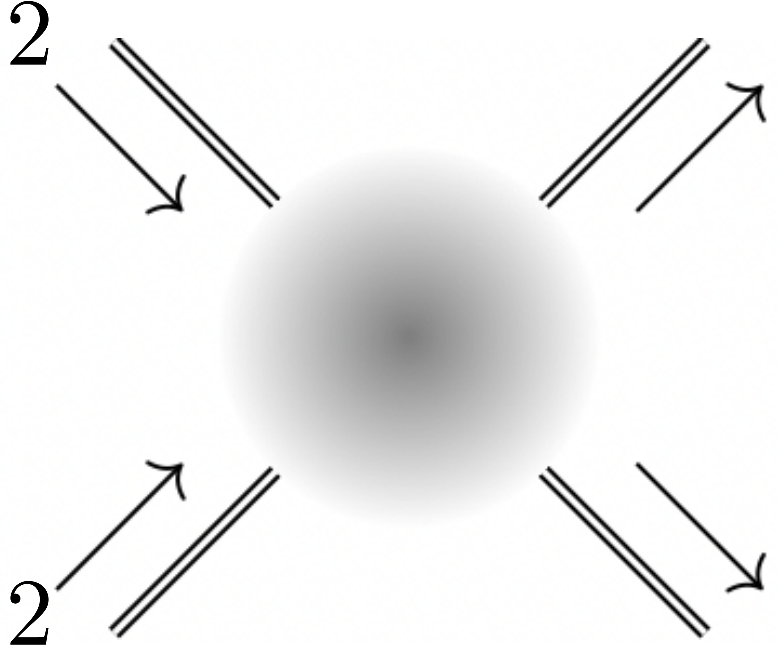


$$= -\frac{i}{m}\frac{\eta^{\mu\nu}}{\omega^2}$$

Generating Function Emerging from WQFT

- Kinematics for the 2->2 scattering process (binary system)

$$p_1 = \bar{p}_1 + q/2 \qquad p'_1 = \bar{p}_1 - q/2$$



$$p_2 = \bar{p}_2 - q/2 \qquad p'_2 = \bar{p}_2 + q/2$$

"Barred vs. unbarred" variables:

$$p_i^2 = p_i'^2 = m_i^2 \,, \ \bar{p}_i^2 = \bar{m}_i^2$$

$$u_i = \frac{p_i}{m_i} \,, \ \bar{u}_i = \frac{\bar{p}_i}{\bar{m}_i}$$

  - In WQFT, the two sets of variables typically correspond to different interpretations of the asymptotic states. The differences however would not enter the generating function.
    [Mogull, Plefka, Steinhoff]

- Generating function identified with the WQFT path integral in the classical limit

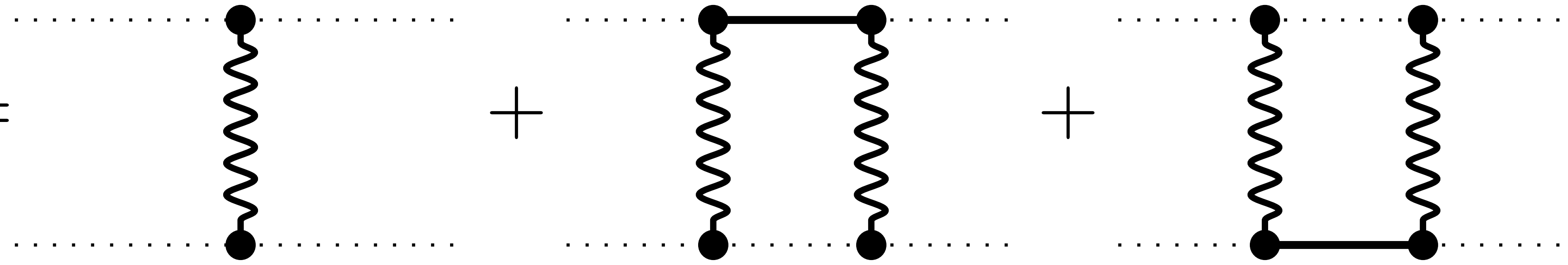
$$e^{i\delta} = \mathcal{Z}_{\mathrm{WQFT}} = \int \mathcal{D}[A] \prod_{j=1}^{2} \mathcal{D}[z_j]\, e^{i\left(\mathcal{S}_{\mathrm{EM}} + \sum_{j=1}^{2} \mathcal{S}_j\right)}$$

  - The phase is a purely classical quantity
  - Identify the phase with connected WQFT diagrams without iterations from lower orders
  - Similar to the eikonal phase and radial action, but not entirely the same

Generating Function at 1PM & 2PM

- At 1PM and 2PM, the phase is given by

$$i\left(\delta^{(0)}+\delta^{(1)}\right)=$$



Impact parameter: the closest distance between the two wordlines

$$=\int_q e^{ib\cdot q}\,\hat{\delta}(q\cdot u_1)\hat{\delta}(q\cdot u_2)\left[-\frac{ie^2q_1q_2\gamma}{q^2}+ie^4q_1^2q_2^2\frac{\big((2D-7)\gamma^2-1\big)(m_1+m_2)}{2(\gamma^2-1)m_1m_2}G_i^{(1)}\right]$$

- Post-Minkowskian expansion: weak coupling (expansion in the coupling constant)
- Dimensional regularization: 1PM phase has a pole which does not affect physical observables
- 2PM: IBP reduction leads to one master integral [LiteRed]

$$G_i^{(1)}=\int_{\ell_1}\frac{\hat{\delta}(\ell_1\cdot u_i)}{\ell_1^2(q-\ell_1)^2}=\frac{(4\pi)^{\epsilon-\frac{3}{2}}\Gamma\left(\frac{1}{2}-\epsilon\right)^2\Gamma\left(\frac{1}{2}+\epsilon\right)}{(-q^2)^{\frac{1}{2}+\epsilon}\Gamma(1-\epsilon)}$$

- Final result:

$$\delta^{(0)}=\alpha q_1q_2\frac{\gamma}{\sqrt{\gamma^2-1}}\frac{\Gamma(-\epsilon)}{\pi^{1-\epsilon}}(\mathsf{b}^2)^\epsilon\,,\qquad \delta^{(1)}=-(\alpha q_1q_2)^2\frac{\pi(m_1+m_2)}{m_1m_2\sqrt{\gamma^2-1}\,\mathsf{b}}$$

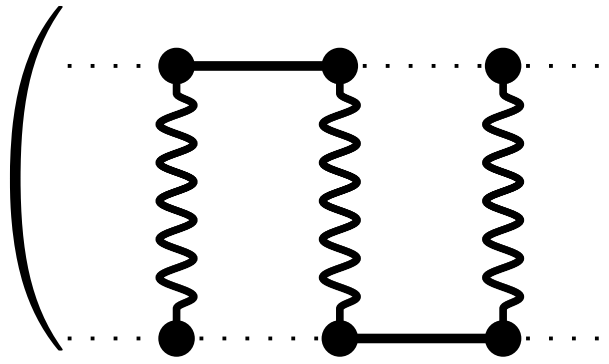
$$\gamma=u_1\cdot u_2$$
$$\mathsf{b}=|b|=\sqrt{-b^2}$$
$$\alpha=e^2/(4\pi)$$

Generating Function at 3PM

- At 3PM, we consider three different sectors: comparable masses, probe limit and radiation reaction

- Comparable masses:

$$i\delta^{(2)}\Big|_{m_1 m_2} = \frac{1}{2!}\Bigg($$



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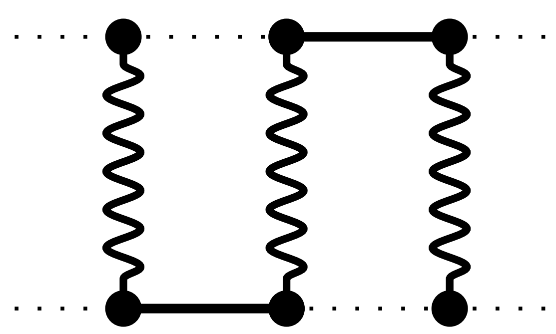


$$\Bigg)$$

$$= \frac{ie^6 q_1^2 q_2^3}{2m_1 m_2}\int_q e^{ib\cdot q}\prod_{i=1}^{2}\hat{\delta}(q\cdot u_i)\int_{\ell_1\ell_2}\frac{\hat{\delta}(\ell_1\cdot u_2)\hat{\delta}(\ell_2\cdot u_1)}{\ell_1^2\ell_2^2(q-\ell_1-\ell_2)^2}$$

$$\left[-\frac{\gamma(q-\ell_2)^2}{(\ell_1\cdot u_1)^2} - \frac{\gamma(q-\ell_1)^2}{(\ell_2\cdot u_2)^2} + \frac{\gamma^3(q-\ell_1)^2(q-\ell_2)^2}{2(\ell_1\cdot u_1)^2(\ell_2\cdot u_2)^2} - \frac{\gamma^2 q^2}{(\ell_1\cdot u_1)(\ell_2\cdot u_2)} - \frac{2(\ell_1\cdot u_1)}{(\ell_2\cdot u_2)} - \frac{2(\ell_2\cdot u_2)}{(\ell_1\cdot u_1)}\right]$$

- IBP reduction leads to 4 master integrals: $G_{0,0,0,0,1,1,1}$, $G_{0,0,0,0,1,2,1}$, $G_{0,0,0,0,2,1,1}$, $G_{1,1,0,0,1,1,1}$

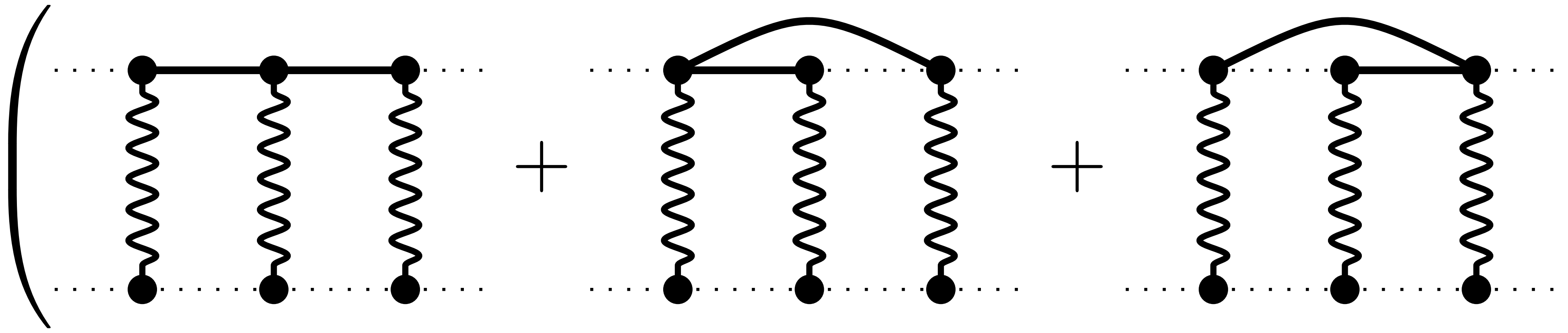
$$G_{n_1 n_2\cdots n_7} = \int_{\ell_1\ell_2}\frac{\hat{\delta}(\ell_1\cdot u_2)\hat{\delta}(\ell_2\cdot u_1)}{\rho_1^{n_1}\rho_2^{n_2}\cdots\rho_7^{n_7}}$$

$$\rho_1 = \ell_1\cdot u_1\,,\quad \rho_2 = \ell_2\cdot u_2\,,\quad \rho_3 = \ell_1^2\,,\quad \rho_4 = \ell_2^2\,,$$
$$\rho_5 = (q-\ell_1-\ell_2)^2\,,\quad \rho_6 = (q-\ell_1)^2\,,\quad \rho_7 = (q-\ell_2)^2$$

Generating Function at 3PM

- Probe limit

$$i\delta^{(2)}\Big|_{m_2^2} = \frac{1}{3!}\Bigg($$



$$\Bigg)$$

$$= \frac{ie^6 q_1^3 q_2^3}{12 m_1^2} \int_q e^{ib\cdot q} \prod_{i=1}^{2} \hat{\delta}(q\cdot u_i) \int_{\ell_1 \ell_2} \frac{\hat{\delta}(\ell_1\cdot u_2)\hat{\delta}(\ell_2\cdot u_2)}{\ell_1^2 \ell_2^2 (q-\ell_1-\ell_2)^2} \Bigg[\frac{\gamma\big((q-\ell_1)^2-q^2\big)}{(\ell_1\cdot u_1)^2} + \frac{2\gamma\big((q-\ell_2)^2-q^2\big)}{(\ell_2\cdot u_1)^2}$$

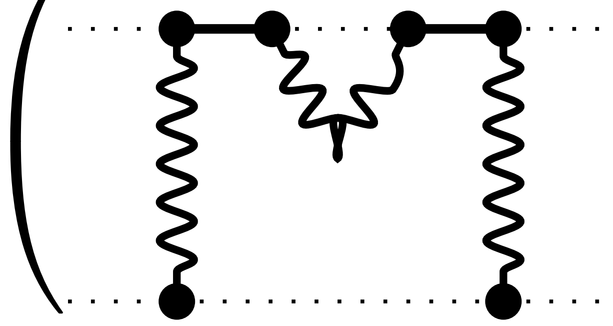
$$+ \frac{\gamma^3 q^2 (q-\ell_2)^2}{(\ell_1\cdot u_1)^2(\ell_2\cdot u_1)^2} - \frac{\gamma^3 (q-\ell_2)^4}{(\ell_1\cdot u_1)^2(\ell_2\cdot u_1)^2} + \frac{\gamma^3 (q-\ell_2)^2\big((q-\ell_2)^2+(q-\ell_1)^2-q^2\big)}{(\ell_1\cdot u_1)^3(\ell_2\cdot u_1)} \Bigg]$$

- Symmetrization: relabelling loop momenta to compensate the artificial "directions" built in the WQFT Feynman rules
- Two probe-limit sectors are related by relabelling.
- IBP reduction leads to only one master integral as well, which is a feature of the probe limit in the gravitational background observed up to five loops! [Brandhuber, Chen, Travaglini, Wen] [Bjerrum-Bohr, Planté, Vanhove]

Generating Function at 3PM

- Radiation reaction

$$i\delta^{(2)}\Big|_{\text{r.r.}} = \frac{1}{2!}\Bigg($$



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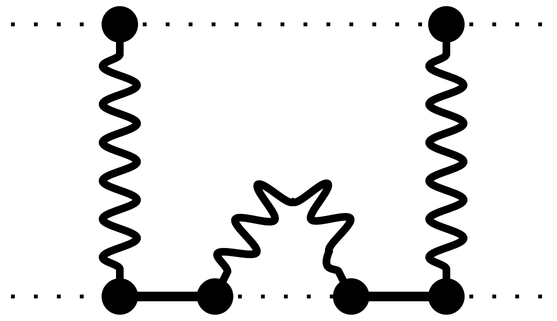

$$\Bigg)$$

$$= \frac{ie^6 q_1^4 q_2^2}{2m_1^2} \int_q e^{ib\cdot q} \prod_{i=1}^{2} \delta(q\cdot u_i) \int_{\ell_1 \ell_2} \frac{\delta(\ell_1\cdot u_2)\delta(\ell_2\cdot u_1)}{\ell_1^2 (q-\ell_1)^2 (q-\ell_1-\ell_2)^2}$$

$$\left[1 + \frac{(\ell_2\cdot u_2)^2}{(\ell_1\cdot u_1)^2} - \frac{\gamma^2 q^2}{2(\ell_1\cdot u_1)^2} + \frac{\gamma \ell_2^2 (\ell_2\cdot u_2)}{(\ell_1\cdot u_1)^3} + \frac{\gamma^2 \ell_2^2 (q-\ell_2)^2}{2(\ell_1\cdot u_1)^4}\right] + \big(\{q_1, m_1\} \leftrightarrow \{q_2, m_2\}\big)$$

  - IBP reduction leads to one master integral $G_{0,0,1,0,1,1,0}$

- In all three sectors, we see that the final integrands are equivalent to those from the HEFT diagrams responsible for the HEFT phase, after IBP reduction
  [Brandhuber, Chen, Travaglini, Wen]

- Due to potential subtleties in the imaginary parts of the master integrals, we restrict our discussions to the real part of this generating function for now

Generating Function at 3PM

- Master integrals
  - Multi-soft expansion: master integrals without all the soft graviton poles can be discarded
  - Standard method for computing the master integrals: differential equation + boundary values
    - Potential region vs. Radiative region
      [Bern et al] [Brandhuber et al]
- Final result for the real part of the 3PM phase

$$\mathrm{Re}\,\delta^{(2)}\Big|_{m_1 m_2} = -\frac{2(\alpha q_1 q_2)^3\,(\gamma^4 - 3\gamma^2 + 3)}{3 m_1 m_2\,\mathsf{b}^2\,(\gamma^2-1)^{5/2}} + \frac{2(\alpha q_1 q_2)^3\gamma^2\left(\gamma\sqrt{\gamma^2-1} - \mathrm{arccosh}\gamma\right)}{m_1 m_2\,\mathsf{b}^2\,(\gamma^2-1)^{5/2}}$$

$$\mathrm{Re}\,\delta^{(2)}\Big|_{m_2^2} = \frac{(\alpha q_1 q_2)^3\,\gamma(2\gamma^2-3)}{m_1^2\,\mathsf{b}^2\,(\gamma^2-1)^{5/2}} \qquad \mathrm{Re}\,\delta^{(2)}\Big|_{\mathrm{r.r.}} = \frac{-2(\alpha q_1 q_2)^3\gamma^2}{3 m_1 m_2\,\mathsf{b}^2\,(\gamma^2-1)}\left(\frac{q_1/m_1}{q_2/m_2} + \frac{q_2/m_2}{q_1/m_1}\right)$$

  - Conservative vs. Radiative
  - Crucial difference from the eikonal phase (no iteration); possible subtle difference from the radial action

Scattering Angle

- Scattering angle from the generating function

$$\chi = -\frac{\partial \delta}{\partial J}$$

$$J = \mathsf{p}\mathsf{b}$$

$$\mathsf{p} = \frac{m_1 m_2}{E}\sqrt{\gamma^2 - 1}$$

$$E = \sqrt{m_1^2 + m_2^2 + 2m_1 m_2 \gamma}$$

- Scattering angle up to 3PM

[Bern, Gatica, Herrmann, Luna, Zeng] [Saketh, Vines, Steinhoff, Buonanno]

$$\chi^{(0)} = \frac{2\alpha q_1 q_2 E\,\gamma}{m_1 m_2\,\mathsf{b}\,\sqrt{\gamma^2 - 1}} \qquad \chi^{(1)} = -\frac{(\alpha q_1 q_2)^2 \pi\, E(m_1 + m_2)}{2m_1^2 m_2^2\,\mathsf{b}^2\,(\gamma^2 - 1)}$$

$$\chi^{(2)}_{\mathrm{con}} = -\frac{4(\alpha q_1 q_2)^3 E\,(\gamma^4 - 3\gamma^2 + 3)}{3m_1^2 m_2^2\,\mathsf{b}^3\,(\gamma^2 - 1)^3} + \frac{2(\alpha q_1 q_2)^3 E(m_1^2 + m_2^2)\,\gamma(2\gamma^2 - 3)}{3m_1^3 m_2^3\,\mathsf{b}^3\,(\gamma^2 - 1)^3}$$

$$\chi^{(2)}_{\mathrm{rad}} = \frac{4(\alpha q_1 q_2)^3 E\,\gamma^2}{m_1^2 m_2^2\,\mathsf{b}^3}\left(\frac{\gamma^3}{(\gamma^2 - 1)^{5/2}} - \frac{\mathrm{arccosh}\gamma}{(\gamma^2 - 1)^3}\right) - \frac{4(\alpha q_1 q_2)^3 E\,\gamma^2}{3m_1^2 m_2^2(\gamma^2 - 1)^{3/2}}\left(\frac{q_1/m_1}{q_2/m_2} + \frac{q_2/m_2}{q_1/m_1}\right)$$

Conclusions & Outlook

- Highly streamlined method for obtaining a generating function from WQFT in the context of scalar-QED
- Scattering angle computed straightforwardly from this generating function
- Generating function coincides with the recently proposed HEFT phase up to 3PM. Precise relation and potential differences between the two quantities remain to be clarified.
- Beyond the toy model: the above is expected to carry over to the gravitational background
- Higher PM orders: possible pathway to an all-loop result in the probe limit
- Further investigation into the connections with the eikonal phase and the radial action