On Galilean conformal field theory

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Galilean conformal field theory

Some basic facts:

- Galilean conformal field theories are non-relativistic field theories with Galilean conformal albebra (GCA) [Bagchi and Gopakumar '09]:

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c_L}{12}(n^2-1)n\delta_{n+m,0} \\ [L_n, M_m] &= (n-m)M_{n+m} + \frac{c_M}{12}(n^2-1)n\delta_{n+m,0} \\ [M_n, M_m] &= 0 \end{aligned}$$

- GCA can in fact be obtained by taking non(ultra)-relativistic limit $\epsilon \to 0$ of the 2d conformal Virasoro algebra:

$$NR: \quad L_n = \mathcal{L}_n + \bar{\mathcal{L}}_n \quad M_n = \epsilon(\mathcal{L}_n - \bar{\mathcal{L}}_n), \qquad c_L = \frac{c+\bar{c}}{12} \quad c_M = \frac{\epsilon(c-\bar{c})}{12}$$

$$UR: \quad L_n = \mathcal{L}_n - \bar{\mathcal{L}}_n \quad M_n = \epsilon(\mathcal{L}_n + \bar{\mathcal{L}}_n), \qquad c_L = \frac{c-\bar{c}}{12} \quad c_M = \frac{\epsilon(c+\bar{c})}{12}$$

- It is isomorphic to the Bondi-Metzner-Sachs (BMS) algebra in 3d, which motivates the study of a holography theory in asymptotic flat spacetime called BMS/GCA correspondence. [Bagchi '10]

Kinematics

The theory of GCFT (or BMSFT) can be developed similarly as 2d CFTs:

- From the GCA symmetry and its representation theory: primaries and descendants.
- Correlation functions of (quasi-)primary operators are severely restricted [Bagchi, Gopakumar, Mandal and Miwa '09; Bagchi, Gary and Zodinmawia '16]:

$$G_2(x_1,x_2,y_1,y_2) = d\delta_{\Delta_1,\Delta_2}\delta_{\xi_1,\xi_2}|x_{12}|^{-2\Delta_1}e^{2\xi_1\frac{y_{12}}{x_{12}}},$$

$$G_3(x_1,x_2,x_3,y_1,y_2,y_3) = c_{123}|x_{12}|^{-\Delta_{123}}|x_{23}|^{-\Delta_{231}}|x_{31}|^{-\Delta_{312}}e^{\xi_{123}\frac{y_{12}}{x_{12}}}e^{\xi_{312}\frac{y_{31}}{x_{31}}}e^{\xi_{231}\frac{y_{23}}{x_{23}}},$$

where d is the normalization factor and c_{123} is the coefficient of 3-pt function.

$$G_4 = \langle\prod_{i=1}^{4}\mathcal{O}_i(x_i,y_i)\rangle = \prod_{i,j}|x_{ij}|^{\sum_{k=1}^{4}-\Delta_{ijk}/3}e^{\frac{y_{ij}}{x_{ij}}\sum_{k=1}^{4}\xi_{ijk}/3}\mathcal{G}(x,y)$$

where x and y are cross ratios:

$$x \equiv \frac{x_{12}x_{34}}{x_{13}x_{24}}, \qquad \frac{y}{x} \equiv \frac{y_{12}}{x_{12}} + \frac{y_{34}}{x_{34}} - \frac{y_{13}}{x_{13}} - \frac{y_{24}}{x_{24}}.$$

Kinematics

- BMS modular invariance and crossing symmetry: bootstrap program in principle. [Bagchi, Detournay, Fareghbal and Simon '12; Jiang, Song and Wen '17; Bagchi, Gary and Zodinmawia '16]
- Similar to LCFTs, there are multiplets (reducible but indecomposable representations) with respect to M_0. [Chen, Hao, Liu and Yu '20]
For example, 2 point functions among operators in a rank-r multiplet become:

$$\langle \mathcal{O}_{k_1}(x_1, y_1)\mathcal{O}_{k_2}(x_2, y_2)\rangle = \begin{cases} 0 & \text{for } q < 0 \\ d_r\, |x_{12}|^{-2\Delta_1} e^{2\xi_1 \frac{y_{12}}{x_{12}}} \frac{1}{q!} \left(\frac{2y_{12}}{x_{12}}\right)^q, & \text{otherwise} \end{cases}$$

where

$$q = k_1 + k_2 + 1 - r,$$

and d is the overall normalization of this rank-r multiplet. Here we denote $\mathcal{O}_{k_i}$ $(k_i = 0, ..., r-1)$ as the $(k_i + 1)$-th operator in the multiplet.
- Degenerate case
For $\xi = 0$: with respect to the global GCA, there are emergent null states. [Chen, Hao, Liu and Yu '22]
For $c_M = 0$: the theory will reduce to be a chiral one, provided the symmetry algebra is not enlarged. [Bagchi, Gopakumar, Mandal and Miwa '09; Hao, Song, Xie and Zhong '21]

Galilean minimal model: a native attempt

- A natural way to construct Galilean minimal models is to calculate its Kac determinant, then find the null states and fusion rules. At level N, it is [Hao '18, unpublished work]:

$$detM_N(c_M,\xi) = (-1)^N [\prod_{ab\leq N, a,b\in\mathbb{N}^+} \chi(a,b)^{\theta(a,b)}]^2$$

where

$$\chi(a,b) = (2a\xi + \frac{c_M}{12}a(a^2-1))^b b!$$

$$\theta(a,b) = \sum_{i=0}^{N-ab} P(i) f(N-ab-1,a)$$

where $P(N)$ is the number of partitions of N. $f(N,a)$ is the number of partitions of N where integer a does not appear in the partition, it can be written as:

$$\sum_{N=0}^{\infty} f(N,a)x^N = \prod_{k\neq a}^{\infty} \frac{1}{1-x^k}$$

- The above Galilean Kac determinant only depends on c_M and ξ, though there are elements which have c_L and(or) Δ dependence in the Gram matrix.

Galilean minimal model: a native attempt

- The null states is determined by the vanishing curve:

$$\chi(a, b) = 0$$

  when these null states are modded out, one can calculate another Kac determinant of the Gram matrix of the remaining states. It turns out that this Kac determinant depends on c_L and Δ, the corresponding vanishing condition is :

$$\Delta + \frac{c_L(a^2 - 1)}{24} = A(a, b) = \text{const}$$

- To find potential minimal models, we need first find the intersection points of vanishing curves, which gives:

$$c_M = 0, \qquad \xi = 0$$

  under this condition, the theory will reduce to a chiral CFT. So we will find the chiral part of the standard Virasoro minimal models.

- As is well known, chiral minimal models can not be a full local theory. A crucial point is the loss of modular invariance of its partition function: in order to be modular invariant, we need its anti-chiral counterpart.

A basic question

Unlike 2d CFT, GCFT is in fact poor in concrete examples:

- As we have shown above, construction of minimal models based on GCA fails.
- The only known GCFT is the free ones, which originate from the study of the tensionless string. [Schild '77; Isberg, Lindstrom, Sundborg and Theodoridis '93]

So a basic question is: does there indeed exist any non-trivial (interacting) GCFTs?

We find it is possible to construct such non-trivial theories by considering enlarged symmetries of the GCA. In particular, we initial the study of rational Galilean conformal field theory (RGCFT), which by definition is the analogue of the Rational conformal field theory but with (enlarged) Galilean conformal symmetry.

Outline

- The free theories: BMS free scalar is studied in detail in [Hao, Song, Xie and Zhong '21]. Here we will first study another free theory: the inhomogenous BMS free fermion, whose action is given in [Bagchi, Banerjee, Chakrabortty and Parekh '17].
- Considering the Ramond sector of the inhomogenous BMS free fermion, we will give a free (fermion) field realization of the Galilean Ising model, which serve as the first example of rational (or minimal) Galilean conformal field theories.
- The Galilean Ising model is not the minimal model with respect to the GCA, instead, its underlying symmetry is a (quantum) W algebra of type $W(2,2,2,1)$, whose classical counterpart had recently appeared in the BMS side and is called the (classical) conformal BMS (CBMS) algebra. [Fuentealba, Gonzalez, Perez, Tempo and Troncoso '20]
  The underlying symmetry of the BMS free scalar can also be a W algebra, whose type is $W(2,2,2,3)$.

Outline

- An important lesson from these 3 theories is the existence of an extra current K of dimension $\Delta = 2$ in the symmetry algebra. In fact, to obtain nontrivial minimal models, K must be included in the BMS algebra.
- From the BMS point of view, K is the 'superspecial conformal transformations' [Fuentealba, Gonzalez, Perez, Tempo and Troncoso '20], (K, T, M) form a triplet with respect to M_0.
- **So we give a proposal for the underlying symmetry of a GCFT (with $c_M = 0$): it will be a W algebra of type $W(2, 2, 2, *)$ and contains a BMS$_3$ subalgebra, where * represent other possible currents.**
- In fact, * can not be empty (trivial). The next simplest case is when * includes only one (bosonic) current. Then $W(2, 2, 2, 1)$ and $W(2, 2, 2, 3)$ is the only 2 possibility, which happen to be the symmetry algebras of the BMS free fermion (or Galilean Ising) and the BMS free scalar respectively.
- Quantum Drinfeld–Sokolov reduction will give us lots of W algebras of type $W(2, 2, 2, *)$, among them those who contains a BMS$_3$ subalgebra will potentially give rational Galilean conformal field theories.

The (inhomogenous) BMS free fermion

- There are 2 different free fermion theories arise from the tensionless (super-)string: homogenous and inhomogenous BMS free fermions. [Bagchi, Banerjee, Chakrabortty and Parekh '16 '17]
- Homogenous: no y dependence, it is in fact simply a theory of 2 real chiral fermions.
- Inhomogenous: the theory has the following Lagrangian

$$\mathcal{L} = \psi_1 \partial_0 \psi_0 + \psi_0 \partial_0 \psi_1 - \psi_0 \partial_1 \psi_0$$

where we will always use the notation: $\partial_0 \equiv \partial_y, \quad \partial_1 \equiv \partial_x$. We have 2 BMS free fermions ψ_1 and ψ_0, they form a rank 2 primary multiplet $(\frac{1}{2}\psi_0, \psi_1)^\top$, with dimensions and boost charge:

$$\Delta = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \qquad \xi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The classical theory

- The equations of motion are:

$$\partial_0 \psi_0 = 0, \qquad \partial_0 \psi_1 = \partial_1 \psi_0$$

from which we have the modes expansion:

$$\psi_0(x) = \sum_n B_n x^{-n-\frac{1}{2}}$$

$$\psi_1(x, y) = \sum_n A_n x^{-n-\frac{1}{2}} - (n + \frac{1}{2}) B_n x^{-n-\frac{3}{2}} y$$

- The stress tensor T^μ_ν is found to be:

$$T^0_0 = -\psi_0 \partial_1 \psi_0, \qquad T^1_1 = \psi_0 \partial_1 \psi_0$$

$$T^0_1 = -\psi_1 \partial_0 \psi_1 - \psi_0 \partial_1 \psi_1, \qquad T^1_0 = 0$$

The classical theory

- It is obvious that:

$$T_0^0 + T_1^1 = 0$$

So there are 2 independent components of the stress tensor. So we can define T and M as follows:

$$T \equiv \frac{1}{2} T_1^0 = -\frac{1}{2}(\psi_1 \partial_0 \psi_1 + \psi_0 \partial_1 \psi_1)$$

$$M \equiv \frac{1}{2} T_0^0 = -\frac{1}{2} \psi_0 \partial_1 \psi_0$$

they have the modes expansion:

$$T(x,y) = \sum_n T_n x^{-n-2} - (n+2) M_n x^{-n-3} y$$

$$M(x,y) = \sum_n M_n x^{-n-2}$$

- It can be checked that:

$$\partial_0 T = \partial_1 M$$

Quantization in the NS sector

- According to the canonical quantization, one can obtain the anti-commutation relations of the modes:

$$\{A_n, A_m\} = \{B_n, B_m\} = 0, \qquad \{A_n, B_m\} = \delta_{n+m,0}$$

- Remember that we are discussing the NS sector, which means $n \in \mathbb{N} + \frac{1}{2}$. So we define the vacuum to be:

$$A_n|0\rangle = B_n|0\rangle = 0, \qquad n > 0$$

  This vacuum is in fact a highest weight vacuum, satisfying:

$$L_n|0\rangle = M_m|0\rangle = 0, \qquad n, m \geq -1$$

- Now the stress tensor operators can be written as normal ordered products:

$$T(x, y) = -\frac{1}{2}(: \psi_1 \partial_0 \psi_1 + \psi_0 \partial_1 \psi_1 :)$$

$$M(x) = -\frac{1}{2} : \psi_0 \partial_1 \psi_0 :$$

OPE of the stress tensor

- Using the anti-commutation relation of the modes, one can calculate the correlator of the fundamental fields:

$$\langle \psi_0(x_1, y_1)\psi_0(x_2, y_2)\rangle = 0$$
$$\langle \psi_0(x_1, y_1)\psi_1(x_2, y_2)\rangle = \frac{1}{x_1 - x_2}$$
$$\langle \psi_1(x_1, y_1)\psi_1(x_2, y_2)\rangle = -\frac{y_1 - y_2}{(x_1 - x_2)^2}$$

  So (ψ_0, ψ_1) indeed form a doublet.

- Then we can using the Wick theorem to write the OPE:

$$\begin{aligned}
M(x_1, y_1)M(x_2, y_2) \sim& 0 \\
T(x_1, y_1)M(x_2, y_2) \sim& \frac{2M(x_2, y_2)}{(x_1 - x_2)^2} + \frac{\partial_x M(x_2, y_2)}{x_1 - x_2} \\
T(x_1, y_1)T(x_2, y_2) \sim& \frac{1}{2(x_1 - x_2)^4} + \frac{2T(x_2, y_2)}{(x_1 - x_2)^2} - \frac{4(y_1 - y_2)M(x_2, y_2)}{(x_1 - x_2)^3} \\
&+ \frac{\partial_x T(x_2, y_2)}{x_1 - x_2} - \frac{(y_1 - y_2)\partial_y T(x_2, y_2)}{(x_1 - x_2)^2}
\end{aligned}$$

Central charge

- Translate into the algebra of the modes, one get the BMS algebra:

$$
\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c_L}{12}(n^2-1)n\delta_{n+m,0} \\
[L_n, M_m] &= (n-m)M_{n+m} + \frac{c_M}{12}(n^2-1)n\delta_{n+m,0} \\
[M_n, M_m] &= 0
\end{aligned}
$$

with central charge $c_L = 1$ and $c_M = 0$. This is the central charge of the BMS free fermion.

The Hilbert space

- The Hilbert space of this theory consist of the following states:

L_0 eigenvalue	states
0	$\|0\rangle$
$\frac{1}{2}$	$A_{-\frac{1}{2}}\|0\rangle, \quad B_{-\frac{1}{2}}\|0\rangle$
1	$A_{-\frac{1}{2}}B_{-\frac{1}{2}}\|0\rangle$
$\frac{3}{2}$	$A_{-\frac{3}{2}}\|0\rangle, \quad B_{-\frac{3}{2}}\|0\rangle$
2	$A_{-\frac{1}{2}}A_{-\frac{3}{2}}\|0\rangle, \quad A_{-\frac{1}{2}}B_{-\frac{3}{2}}\|0\rangle, \quad B_{-\frac{1}{2}}A_{-\frac{3}{2}}\|0\rangle, \quad B_{-\frac{1}{2}}B_{-\frac{3}{2}}\|0\rangle$
...	...

- To find the module decomposition of this Hilbert space, we need to find the primary fields. It turns out that there are 4 BMS primaries, all of them have boost charge $\xi = 0$:
  * singlet $\mathbf{1}$ with dimension $\Delta = 0$, the corresponding state is $|0\rangle$.
  * doublet (ψ_0, ψ_1) with dimension $\Delta = \frac{1}{2}$, the corresponding states are $A_{-\frac{1}{2}}|0\rangle$ and $B_{-\frac{1}{2}}|0\rangle$.
  * singlet $\epsilon \equiv: \psi_0\psi_1 :$, with dimension $\Delta = 1$, the corresponding state is $B_{-\frac{1}{2}}A_{-\frac{1}{2}}|0\rangle$.
- Note that though ϵ is a singlet, it has non-trivial y dependence.

The enlarged symmetry

There are important lessons one can learn from the module decomposition of the Hilbert space:

- To organize all the states into representations, one will find many non-primary states, such as $A_{-\frac{1}{2}}A_{-\frac{3}{2}}|0\rangle$, which are not descendants of any primary states. In fact, this is similar with the case of BMS free scalar [Hao, Song, Xie and Zhong '21]. As a result, we need an extra quasi-primary operator (state) with dimension $\Delta = 2$ which enlarge the BMS module:

$$K(x, y) \equiv - : \psi_1 \partial_1 \psi_1 :$$

The corresponding state is just:

$$|K\rangle = -A_{-\frac{1}{2}}A_{-\frac{3}{2}}|0\rangle$$

In fact, it is just the existence of this operator that prevent the decoupling of the M_n (to be null) in the (enlarged) BMS module. This K operator, together with T, M form a stress tensor triplet, with dimension $\Delta = 2$, boost charge $\xi = 0$.

The enlarged symmetry

- Operators in BMS fermions can not be classified into **disjoint** BMS conformal families.

  An example is the stress tensor $|M\rangle = M(0)|0\rangle = M_{-2}|0\rangle$, which is the a descendant of the vacuum $|0\rangle$ but at the same time is a (global) descendant of the primary operator ϵ:

  $$2M(x) = \partial_y \epsilon(x, y)$$

  This fact reflect that the real underlying symmetry is not the GCA but is an enlarged one, the conformal BMS (CBMS), which is a W algebra of type $W(2, 2, 2, 1)$.
- Besides the operator K, CBMS in addition include an current ϵ with $\Delta = 1$. Then $|0\rangle$ and $|\epsilon\rangle$ are in the same CBMS module, so all operators can be classified into **disjoint** CBMS modules.
- CBMS with central charge $c = 1$ is in fact the bosonic subalgebra of the algebra of complex fermions. We will derive it from the bottom up later.

Operators in the stress tensor multiplet

- We have the following operators with $\Delta = 2$:

$$M(x) = -\frac{1}{2}\phi_0\partial_1\psi_0, \quad T(x,y) = -\frac{1}{2}(\psi_1\partial_0\psi_1 + \psi_0\partial_1\psi_1) \quad K(x,y) = -\psi_1\partial_1\psi_1$$

as well as

$$\partial_1\epsilon(x,y) = -\frac{1}{2}(\partial_1\psi_0\psi_1 + \psi_0\partial_1\psi_1) = -\frac{1}{2}(-\psi_1\partial_0\psi_1 + \psi_0\partial_1\psi_1)$$

Note that $\partial_0\epsilon = M$, $\partial_0(\partial_1\epsilon) = \partial_1 M$, $\partial_0 T = \partial_1 M$ and $N(\epsilon\epsilon) = T$.

- The states corresponds to the stress tensor multiplet operators are:

$$\begin{aligned}
|K\rangle &= -A_{-\frac{1}{2}}A_{-\frac{3}{2}}|0\rangle \\
|T\rangle &= -\frac{1}{2}(A_{-\frac{1}{2}}B_{-\frac{3}{2}} + B_{-\frac{1}{2}}A_{-\frac{3}{2}})|0\rangle \\
|M\rangle &= -\frac{1}{2}B_{-\frac{1}{2}}B_{-\frac{3}{2}}|0\rangle \\
|\partial_1\epsilon\rangle &= -\frac{1}{2}(-A_{-\frac{1}{2}}B_{-\frac{3}{2}} + B_{-\frac{1}{2}}A_{-\frac{3}{2}})|0\rangle
\end{aligned}$$

One can check the action of L_0 and M_0:

$$L_0|K\rangle = 2|K\rangle, \quad L_0|T\rangle = 2|T\rangle, \quad L_0|\partial_1\epsilon\rangle = 2|\partial_1\epsilon\rangle, \quad L_0|M\rangle = 2|M\rangle$$

$$M_0|K\rangle = 2|T\rangle - |\partial_1\epsilon\rangle, \quad M_0|T\rangle = 2|M\rangle, \quad M_0|\partial_1\epsilon\rangle = |M\rangle, \quad M_0|M\rangle = 0$$

Operators in the stress tensor multiplet:

- So the stress tensor operators include a triplet $|\mathbb{T}^3\rangle$ and a singlet $|\mathbb{T}^1\rangle$:

$$|\mathbb{T}^3\rangle = (\frac{3}{2}|M\rangle, |T\rangle - \frac{1}{2}|\partial_1\epsilon\rangle, \frac{1}{2}|K\rangle)^\top, \qquad |\mathbb{T}^1\rangle = |T\rangle - 2|\partial_1\epsilon\rangle$$

with conformal dimensions and boost charges:

$$\Delta_{\mathbb{T}^3} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad \Delta_{\mathbb{T}^1} = 2, \qquad \xi_{\mathbb{T}^3} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \xi_{\mathbb{T}^1} = 0$$

It can be easily checked that $|K\rangle$, $|T\rangle$, $|M\rangle$ are Viraroso quasi-primaries ($|K\rangle$, $|M\rangle$ are Viraroso primaries):

$$L_1|K\rangle = L_1|T\rangle = L_1|M\rangle = 0$$

Note that $|M\rangle$ is a global BMS descendant of $|\epsilon\rangle$ but not as a global Viraroso descendant, so it can be a Viraroso quasi-primary. $|\partial_1\epsilon\rangle$ is the global Viraroso descendant of $|\epsilon\rangle$ so is not a Viraroso quasi-primary ($L_1|\partial_1\epsilon\rangle = 2|\partial_1\epsilon\rangle$) With respect to the BMS symmetry, $|T\rangle$ and $|M\rangle$ are BMS quasi-primaries:

$$M_1|T\rangle = M_1|M\rangle = L_1|T\rangle = L_1|M\rangle = 0$$

however, $|K\rangle$ is not a BMS quasi-primary:

$$M_1|K\rangle = -2|\epsilon\rangle$$

Finally, we note that $|\partial_1\epsilon\rangle$ is annihilated by M_1:

$$M_1|\partial_1\epsilon\rangle = M_1L_{-1}|\epsilon\rangle = 2M_0|\epsilon\rangle = 0$$

Novel currents realization of W algebras

- All OPEs and correlators among these (quasi-)primaries can be easily written down using the Wick theorem.
- The OPE among operators in the vacuum module in principle gives the underlying symmetry. However, this will not be a 'good' way to organize the underlying symmetry because some 'currents' in these OPE will not be Virasoro (quasi-)primaries. An example is:

$$K^1(x) \equiv \partial_y K(x,y)$$

  which will appear in the OPE of $M(x)K(x,y)$. So in this way the underlying W algebra is hard to see. Or we can say that GCFT will always gives a novel currents realization of its symmetry algebra.
- Because $\partial_y \Leftrightarrow M_{-1}$, we have a following simple translation rule for7 the currents. Given a current $W^0(z)$ in the chiral algebra, we denote (due to state-operator correspondence):

$$W^0(z) \leftrightarrow |W\rangle, \qquad W^1(z) \leftrightarrow M_{-1}|W\rangle, \quad \dots \quad , W^n(z) \leftrightarrow M_{-1}^n|W\rangle$$

  Then the corresponding current in GCFT will be:

$$G(x,y) = W^0(x) + W^1(x)y + \dots + W^n(x)\frac{y^n}{n!}$$

  It is easy to see that $W^i(z)$ is generally not Virasoro primaries.
- So the current realization of the W(2,2,2,1) and W(2,2,2,3) in terms of BMS free fermion and scalar is different form the one in CFT.

Free fermion realization of Galilean Ising model

- The reason why the above attempt for minimal models fail is that all M_ns decouple from the BMS module, so one may try to start from an enlarged BMS algebra, such that M_ns do not decouple.
- In fact, these enlarged BMS algebras indeed appear in the free scalar and fermion theory, namely, W(2,2,2,1) and W(2,2,2,3).
- As is well-known, Ising model can be represented by the Majorana fermion as follows (fermion-boson duality):

$$\mathbf{1} = \mathbf{1}_{\text{free fermion}}, \quad \epsilon = i : \psi\bar{\psi} :, \quad \sigma = \text{twist operator for the Ramond sector}$$

We will try to find a similar free theory realization to construct the Galilean Ising model.

The Ramond sector

- We discuss the R sector to find the 'spin operator' σ. In the R sector, the modes number $n \in \mathbb{Z}$ so the fermions satisfy:

$$\psi_0(e^{2\pi i}x) = -\psi_0(x), \qquad \psi_1(e^{2\pi i}x, y) = -\psi_1(x, y)$$

  The R sector vacuum is created by a twist operator σ, which will be identified with the 'spin operator' in the Galiean Ising model. The dimension and boost charge of this twist operator can be calculated by considering the expectation value of the stress-energy tensor in the R sector vacuum $|0\rangle_R \equiv \sigma(0,0)|0\rangle$:

$${}_R\langle T(x,y)\rangle_R, \qquad {}_R\langle M(x)\rangle_R$$

- First, we need to calculate the 2-pt function of the ψ_0 and ψ_1 in the R sector, using the commutation relation of the modes, we have :

$${}_R\langle \psi_0(x_1)\psi_0(x_2)\rangle_R = 0$$

$${}_R\langle \psi_0(x_1,y_1)\psi_1(x_2,y_2) + \psi_1(x_1,y_1)\psi_0(x_2,y_2)\rangle_R = \frac{\sqrt{\frac{x_1}{x_2}} + \sqrt{\frac{x_2}{x_1}}}{x_1 - x_2}$$

  In the $x_1 \to x_2$ limit, it coincides with the corresponding one in the NS sector, because short distance behavior is independent of the global boundary conditions.

The Ramond sector

- Because σ is a singlet primary operator, its OPE with stress tensors are:

$$T(x,y)\sigma(0,0)|0\rangle \sim \frac{\Delta\sigma(0,0)}{x^2}|0\rangle + \frac{2y\xi\sigma(0,0)}{x^3}|0\rangle + \dots$$
$$M(x,y)\sigma(0,0)|0\rangle \sim \frac{\xi\sigma(0,0)}{x^2}|0\rangle + \dots$$

- From its definition, the stress tensor can be realized as:

$$T(x,y) = -\frac{1}{2}[\psi_1(x,y)\partial_z\psi_0(z) + \psi_0(x)\partial_z\psi_1(z,w) + \frac{2}{(x-z)^2}]_{x\to z, y\to w}$$

$$M(x) = -\frac{1}{2}[\psi_0(x)\partial_z\psi_0(z)]_{x\to z}$$

let $x - z = \epsilon$, in the $\epsilon \to 0$ limit, they will give the expectation value of the stress-energy tensor in the R sector vacuum:

$$_R\langle M(x)\rangle_R = 0 \qquad _R\langle T(x,y)\rangle_R = \frac{1}{8}\frac{1}{x^2}$$

from the OPE, we find the quantum number of the twist field:

$$\Delta_\sigma = \frac{1}{8}, \qquad \xi_\sigma = 0$$

Note that these values agree with the one from the non-relativistic limit of the Ising model.

The vacuum structure

- Finally, we need to discuss the vacuum structure. Consider the zero modes: A_0 and B_0, as well as the fermionic number operator $(-1)^F$. Recall that:

$$\{A_n, A_m\} = 0, \qquad \{B_n, B_m\} = 0 \qquad \{A_n, B_m\} = \delta_{n+m,0}$$

  We combine them as:

$$C_n = \frac{1}{\sqrt{2}}(A_n + B_n), \qquad D_n = \frac{i}{\sqrt{2}}(A_n - B_n)$$

  it is easy to verify that they satisfy:

$$\{C_n, C_m\} = \delta_{n+m,0}, \qquad \{D_n, D_m\} = \delta_{n+m,0} \qquad \{C_n, D_m\} = 0$$

- As a result, we have transformed the basis of the zero modes of A_0 and B_0 in terms of C_0 and D_0, they obey the clifford algebra just as ψ_0 and $\bar{\psi}_0$:

$$C_0^2 = \frac{1}{2}, \qquad D_0^2 = \frac{1}{2}, \qquad \{C_0, D_0\} = 0$$

  so they can be realized in terms of Pauli matrices:

$$C_0 = \frac{\sigma_x + \sigma_y}{2}(-1)^{\sum_{n>0} C_{-n}C_n + D_{-n}D_n}$$

$$D_0 = \frac{\sigma_x - \sigma_y}{2}(-1)^{\sum_{n>0} C_{-n}C_n + D_{-n}D_n}$$

$$(-1)^F = \sigma_z(-1)^{\sum_{n>0} C_{-n}C_n + D_{-n}D_n}$$

The vacuum structure

- Now we have 2 twist fields: σ and μ, they create 2 Ramond sector vacuums, they transform under the zero modes as:

$$C_0|\sigma\rangle = \frac{1-i}{2}|\mu\rangle, \qquad C_0|\mu\rangle = \frac{1+i}{2}|\sigma\rangle$$
$$D_0|\sigma\rangle = \frac{1+i}{2}|\mu\rangle, \qquad D_0|\mu\rangle = \frac{1-i}{2}|\sigma\rangle$$

  in terms of A_0 and B_0, we have:

$$A_0|\sigma\rangle = \frac{1-i}{\sqrt{2}}|\mu\rangle, \qquad A_0|\mu\rangle = 0$$
$$B_0|\sigma\rangle = 0, \qquad A_0|\mu\rangle = \frac{1+i}{\sqrt{2}}|\sigma\rangle$$

- So combine the states built on these two R vacuums $|\sigma\rangle$ and $|\mu\rangle$, we can check that there are two representations of the CBMS (not the GCA!), both with $\Delta = \frac{1}{8}$ and $\xi = 0$:
  - even number of fermions built on $|\sigma\rangle$ + odd number of fermions built on $|\mu\rangle$, the primary state is $|\sigma\rangle$.
  - odd number of fermions built on $|\sigma\rangle$ + even number of fermions built on $|\mu\rangle$, the primary state is $|\mu\rangle$.

The fusion rules

- Next we can calculate the fusion rules, which turns out to be:

$$\begin{array}{ccc} [\epsilon][\epsilon]=[\mathbf{1}], & [\Psi][\Psi]=[\mathbf{1}]+[\epsilon] & . \\ [\sigma][\sigma]=[\mathbf{1}]+[\epsilon] & [\mu][\mu]=[\mathbf{1}]+[\epsilon], & [\sigma][\mu]=[\Psi] \\ [\sigma][\epsilon]=[\sigma], & [\mu][\epsilon]=[\mu], & . \\ [\Psi][\sigma]=[\mu], & [\Psi][\mu]=[\sigma], & [\Psi][\epsilon]=[\Psi] \end{array}$$

  where we denote the doublet representation of the fundamental fields (ψ_0, ψ_1) as Ψ.

- This fusion algebra has a closed subalgebra including only: $\mathbf{1}$, σ and ϵ (another equivalent one includes $\mathbf{1}$, μ and ϵ)

$$\begin{array}{ccc} [\epsilon][\epsilon]=[\mathbf{1}], & [\sigma][\sigma]=[\mathbf{1}]+[\epsilon] & [\sigma][\epsilon]=[\sigma] \\ [\mathbf{1}][\epsilon]=[\epsilon], & [\mathbf{1}][\sigma]=[\sigma], & [\mathbf{1}][\mathbf{1}]=[\mathbf{1}] \end{array}$$

  This fusion algebra should be understood with respect to the CBMS, rather than the GCA. So the fusion rule of the Galilean Ising model can be written more suitably as follows ('C' denote conformal BMS):

$$[\mathbf{1}]_{\mathrm{C}}[\mathbf{1}]_{\mathrm{C}}=[\mathbf{1}]_{\mathrm{C}}, \quad [\mathbf{1}]_{\mathrm{C}}[\sigma]_{\mathrm{C}}=[\sigma]_{\mathrm{C}}, \quad [\sigma]_{\mathrm{C}}[\sigma]_{\mathrm{C}}=[\mathbf{1}]_{\mathrm{C}}$$

- All the structure constants can also be easily read off from this free field realization (in fact, they are trivial).

Partition function

- The partition function for the Galilean Ising model or the BMS free fermion is:

$$
\begin{aligned}
Z(\tau,\rho) &\equiv \mathrm{Tr}_{\mathcal{H}} e^{2\pi i\tau(L_0-\frac{c_L}{24})+2\pi i\rho(M_0-\frac{c_M}{24})} \\
&= \frac{\theta_2(\tau)+\theta_3(\tau)+\theta_4(\tau)}{2\eta(\tau)} \\
&= \chi_0^{(W)}(\tau)+\chi_{\frac{1}{8}}^{(W)}(\tau) = \chi_0^2(\tau)+\chi_{\frac{1}{2}}^2(\tau)+\chi_{\frac{1}{16}}^2(\tau)
\end{aligned}
$$

  where $\chi_\Delta^{(W)}$ is the character of the W-module. This partition function is BMS modular invariant.

- So we have the following relations:

$$\text{Ising model} \xrightarrow{\text{NR limit}} \text{Galilean Ising} \xleftarrow[\text{modular invariant}]{\text{chiral}} \text{Free boson with } r=1$$

  or (by the fermion-boson duality):

$$\text{Majorana fermion} \xrightarrow{\text{NR limit}} \text{BMS free fermion} \xleftarrow[\text{modular invariant}]{\text{chiral}} \text{Dirac fermion}$$

  In the arrow $\rightarrow$, the the underlying symmetry have been changed. In the arrow $\leftarrow$, in order to be BMS modular invariant, only part of the chiral theory (the chiral W(2,2,2,1) minimal model) is allowed.

Bottom up construction of W(2,2,2,1)

- We will now show the explicit form of the underlying symmetry algebra of BMS free fermion or Galilean Ising, namely, the (quantum) CBMS.
- These theories have central charge $c = 1$. Because we are interested in RGCFTs, so we want to derive the type W(2,2,2,1) algebra from bottom up to see whether the central charge can be deformed (generic W algebra).
- The classical W(2,2,2,1) algebra, as a conformal extension of the BMS algebra, is obtained in [Fuentealba, Gonzalez, Perez, Tempo and Troncoso '20]. Conformal means:

  $$L_n : \text{superrotations}, \qquad M_n : \text{supertranslations}$$

  $$\epsilon_n : \text{superdilatations}, \qquad K_n : \text{superspecial conformal transformations}$$

  it is shown there that the classical BMS is a non-linear and generic classical W algebra.
- So as a byproduct, we will obtain its quantum version.

Bottom up construction of W(2,2,2,1)

- We start with the BMS_3 (with $c_M = 0$) and impose $K_0 \equiv M_0^\dagger$, so obtain:

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c_L}{12}(n^2-1)n\delta_{n+m,0} \\
[L_n, M_m] &= (n-m)M_{n+m} \\
[L_n, K_m] &= (n-m)K_{n+m} \\
[M_n, M_m] &= 0 \\
[K_n, K_m] &= 0
\end{aligned}$$

  If we recombine K_n and M_n as

$$I_n \equiv K_n + M_n, \qquad J_n \equiv i(K_n - M_n)$$

  then $I_n^\dagger = I_{-n}$, $J_n^\dagger = J_{-n}$, they are ordinary Virasoro primaries with dimension $\Delta = 2$.

- Including a current ϵ of dimension 1 into the algebra, so:

$$[L_n, \epsilon_m] = -m\epsilon_{n+m}, \quad [\epsilon_n, \epsilon_m] = kn\delta_{n+m,0}$$

  Jacobi identity restrict the following 2 commutator as:

$$[K_n, \epsilon_m] = K_{n+m}, \quad [M_n, \epsilon_m] = -M_{n+m}$$

- Note that in fact the presence of ϵ gives $c_M = 0$ (Jacobi identity).

Bottom up construction of W(2,2,2,1)

- The remaining commutator is:

$$[M_n, K_m] = (n-m)L_{n+m} + \frac{c(m^2+n^2-nm-1)}{12k}\epsilon_{n+m} + \frac{(4k-c)(n-m)}{4k(c-1)}\Lambda(\epsilon\epsilon)_{n+m} + \frac{2c}{k(c+2)}N(\epsilon L)_{n+m} + \frac{c(12k-c-2)}{2k^2(c+2)(c-1)}\Lambda(\epsilon\epsilon\epsilon)_{n+m} + \frac{c}{2}\frac{n(n^2-1)}{6}\delta_{n+m,0}$$

  where

$$\Lambda(\epsilon\epsilon) \equiv L - \frac{c}{2k}N(\epsilon\epsilon)$$

$$\Lambda(\epsilon\epsilon\epsilon) \equiv N(\epsilon L) - \frac{c+2}{6k}N(\epsilon\epsilon\epsilon)$$

  are quasi-primaries.

- consider all the Jacobi identities, we find:

$$k = \frac{1}{4}, \quad c = 1$$

  As expected, these values are just the ones in the Galilean Ising model.

Bottom up construction of W(2,2,2,1)

- In fact, for $c = 1$ and $k = \frac{1}{4}$, the above commutator simplify much:

$$[M_n, K_m] = 2(n-m)N(\epsilon\epsilon) + \frac{m^2 + n^2 - nm - 1}{3}\epsilon_{n+m} + \frac{16}{3}N(\epsilon\epsilon\epsilon)_{n+m} + \frac{n(n^2-1)}{12}\delta_{n+m,0}$$

One can see above that L (as well as $N(\epsilon L)$) decouple from the algebra. In fact, because $c = 1$ is identical with the central charge of the one from the $u(1)$ Sugawara construction:

$$T = \frac{1}{2k}N(\epsilon\epsilon)$$

so $\Lambda(\epsilon\epsilon)$ and $\Lambda(\epsilon\epsilon\epsilon)$ are in fact null states. Modding out these null states, we effectively decouple L and $N(\epsilon L)$ from the W-algebra.

From the quantum Drinfeld-Sokolov reduction

- From the QDS point of view,

$$sl_2 \xrightarrow[embedding]{\text{non-principal}} B_2 \Rightarrow W(2,2,2,1)(\text{generic c}) \subseteq \text{BMS free fermion (c=1)}$$

$$sl_2 \xrightarrow[embedding]{\text{non-principal}} G_2 \Rightarrow W(2,2,2,3)(\text{generic c}) \subseteq \text{BMS free scalar (c=2)}$$

- With BMS_3 embedded:

$\text{BMS}_3 \hookrightarrow W(2,2,2,1) \rightsquigarrow$ only for c=1, 'exotic' W algebra

$\text{BMS}_3 \hookrightarrow W(2,2,2,3) \rightsquigarrow$ only for $c = 2$/generic, to be determined.

- Classical vs Quantum: BMS_3 can always be embedded into the classical CBMS, namely, in the classical $W(2,2,2,1)$ one have [Fuentealba, Gonzalez, Perez, Tempo and Troncoso '20]:

$$i\{M_n, M_m\} = 0, \qquad i\{K_n, K_m\} = 0$$

However, BMS_3 can only be embedded into the quantum CBMS when $c = 1$. For other c, we have:

$$[M_n, M_m] \equiv X_{n+m} \neq 0, \qquad [K_n, K_m] \equiv Y_{n+m} \neq 0$$

In the (quasi-)classical limit, the terms X_{n+m} and Y_{n+m} vanish.

Outlook

- Having the Galilean Ising model at hand, we are mostly interested in bootstrapping other possible rational Galilean conformal field theories. We have proposed that the underlying symmetry will be a W algebra of type W(2,2,2,*) which contains a BMS subalgebra.
- It is not clear whether there are generic W algebras (exist for generic c) of the above type. We are currently constructing W(2,2,2,3) to see whether it become 'exotic' by imposing a BMS_3 embedding. Generally, one can test these embedding in the framework of the quantum Drinfeld-Sokolov reduction.
- Other W algebras which can be obtained directly include the one coming from the BMS ghost system and the supersymmetric version of the conformal BMS algebra (SCBMS). SCBMS is of type $W(2,2,2,\frac{3}{2},\frac{3}{2},1)$, and the classical version was worked out in [Fuentealba, Gonzalez, Perez, Tempo and Troncoso '20]. One may similarly work out the corresponding Galilean minimal model(s).
- Another possible way to obtain RGCFTs is to take "NR limit" of known RCFTs. While in the Galilean Ising model this is achieved by changing the underlying symmetry:

$$\mathrm{Vir} \times \mathrm{Vir} \to W(2,2,2,1)$$

  It is not clear whether this works for general rational (minimal) models.

Outlook

- Recall that (chiral) minimal models can be obtained as:

$$\frac{\hat{su}(2)_k \times \hat{su}(2)_1}{\hat{su}(2)_{k+1}}$$

  So one may also try to construct RGCFTs from the coset construction.
- Notice that rational Galilean conformal field theories may also have primary multiplets, these theories will be similar with logarithmic minimal models. Recalled that the simplest logarithmic minimal model: the so-called triplet model $\mathfrak{W}(1,2)$ [Gaberdiel and Kausch '98], which can be compared with Galilean Ising as:

$$\text{Symplectic fermion} \xrightarrow[\text{subalgebra}]{\text{bosonic}} \mathfrak{W}(1,2) \to \text{W(2,3,3,3) with } c=-2 \text{ (exotic)}$$

$$\text{BMS free fermion} \xrightarrow[\text{subalgebra}]{\text{bosonic}} \text{CMBS} \to \text{W(2,2,2,1) with } c=1 \text{ (`exotic')}$$

  Recall that other logarithmic minimal models $\mathfrak{W}(p,q)$ are also built based on exotic W algebras (of different types). So even the W algebras given by QDS become 'exotic' by imposing a BMS_3 embedding, one may also find the corresponding minimal models.

Thanks for your attention!