On the real-time evolution of pseudo-(Rényi) entropy in 2d CFTs

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Based on arXiv: 2206.11818 & on-going work

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Outline

- ☐ **Introduction of EE and Psuedo entanglement entropy**
- ☐ **Psuedo Renyi entropy in 2D CFT**

1. **Replic trick and setup**
2. **Psuedo Renyi entropy in local quench**
3. **Hidden symmetry**

- ☐ **Summary**

Part 1. Pseudo-(Rényi) Entropy Introduction

Def. Of EE in discrete systems

Divide a quantum system into two parts A and B.

$$H_{tot} = H_A \otimes H_B$$ **Factorization Property**

Example: Spin Chain

Reduced Density Matrix:

$$\rho_A = \mathrm{Tr}_B \rho_{tot}$$

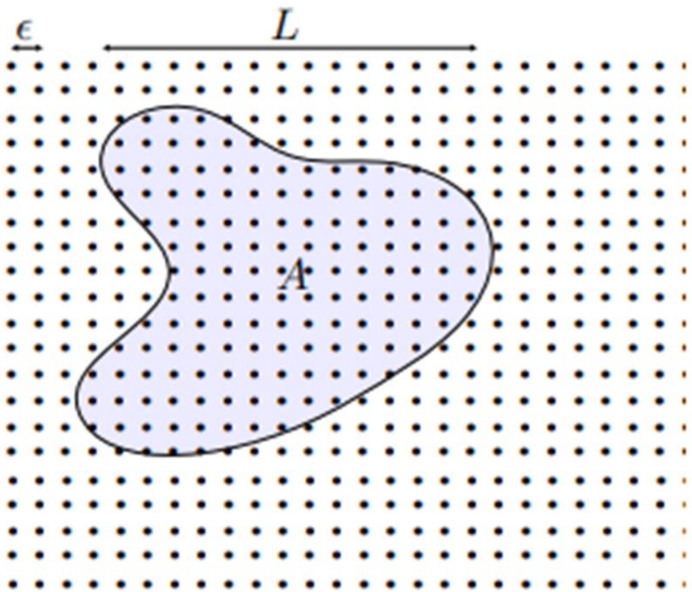
von-Neumann entropy :

Fine grained Entropy

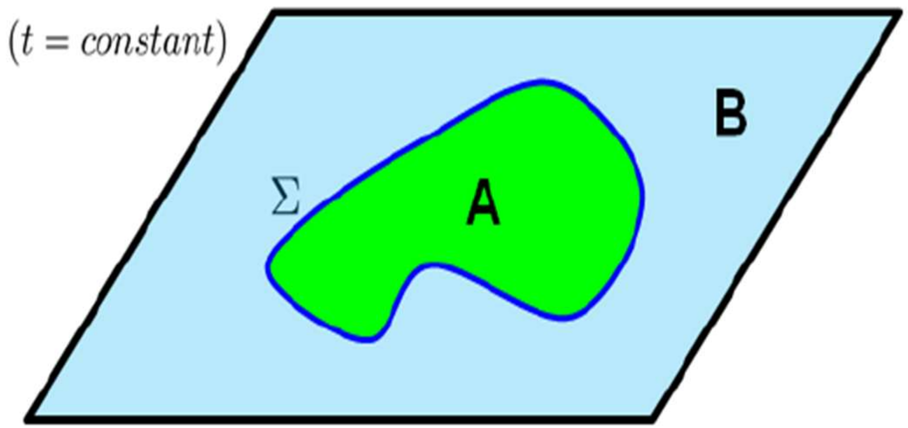
$$S_A = -\mathrm{Tr}_A \, \rho_A \log \rho_A$$

Definition of EE in QFT:

In QFTs, the EE is defined geometrically (called geometric entropy).



Continuum Limit $\epsilon \to 0$



$$S_A = -\mathrm{Tr}_A \rho_A \log \rho_A$$

$$H_{tot} = H_A \otimes H_B$$

No factorization in Gauge theory and gravity!!

Definition of Transition matrix in QFT:

$$\mathcal{T}^{\psi|\varphi} \equiv \frac{|\psi\rangle\langle\varphi|}{\langle\varphi|\psi\rangle}$$

Phi != Psi

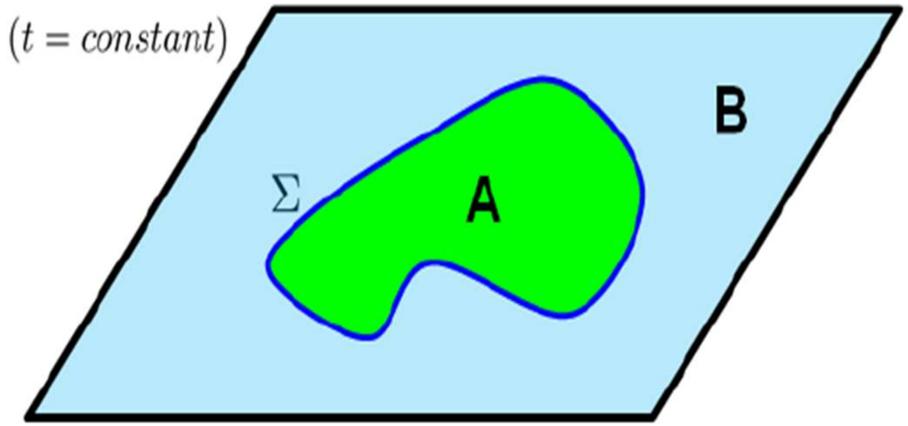
Properties:

$$\mathrm{Tr}\,\mathcal{T}^{\psi|\varphi} = 1$$

$$\left(\mathcal{T}^{\psi|\varphi}\right)^n = \mathcal{T}^{\psi|\varphi}, \quad \forall n \in \mathbb{N}^+$$

$$\mathrm{Tr}\left(\mathcal{T}^{\psi|\varphi}\right)^n = 1$$

$$\mathcal{T}^{\psi|\varphi} = \left(\mathcal{T}^{\varphi|\psi}\right)^\dagger$$



$$H_{tot} = H_A \otimes H_B$$

Reduced Transition matrix:

$$\mathcal{T}_A^{\psi|\varphi} = \mathrm{Tr}_B\left(\mathcal{T}^{\psi|\varphi}\right)$$

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Introduction: Pseudo-(Rényi) entropy

$$\mathcal{T}^{\psi_1|\psi_2} = \frac{|\psi_1\rangle\langle\psi_2|}{\langle\psi_2|\psi_1\rangle}$$

Partial trace

$$\mathcal{T}_A^{\psi_1|\psi_2} = \mathrm{Tr}_{A^c}[\mathcal{T}^{\psi_1|\psi_2}]$$

Transition matrix ($\langle\psi_2|\psi_1\rangle \neq 0$)

Reduced transition matrix

PE: $$S_A = -\mathrm{Tr}[\mathcal{T}_A^{\psi_1|\psi_2} \log \mathcal{T}_A^{\psi_1|\psi_2}]$$

We can also define the corresponding "pseudo-Rényi entropy (PRE)" with respect to $\mathcal{T}_A^{\psi_1|\psi_2}$

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PRE: $$S_A^{(n)} = \frac{1}{1-n} \log \mathrm{Tr}[(\mathcal{T}_A^{\psi_1|\psi_2})^n] \qquad n \in \mathbb{R}^+ \setminus \{1\}$$

$$\lim_{n\to 1} S_A^{(n)} = S_A \quad ✓$$

PE and PRE are normally complex!

Trace of $\left(\mathcal{T}_A^{\psi_1|\psi_2}\right)^n$

$\mathcal{T}_A^{\psi_1|\psi_2}$ is always similar to an upper triangular matrix X_A (Schur's theorem).

$$X_A = U^{-1}\mathcal{T}_A^{\psi_1|\psi_2}U = \begin{pmatrix} \lambda_1 & & & & & * \\ & \ddots & & & & \\ & & \lambda_1 & & & \\ & & & \ddots & & \\ & \mathbf{0} & & & \lambda_m & \\ & & & & & \ddots \\ & & & & & & \lambda_m \end{pmatrix}$$

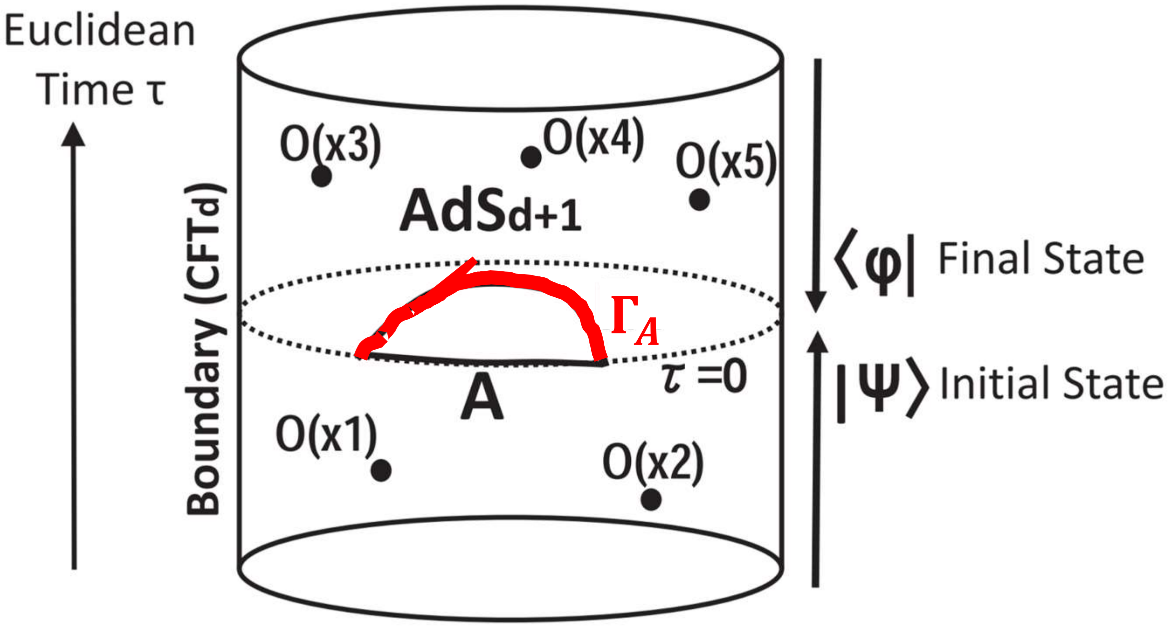
Eigen values of $\mathcal{T}_A^{\psi_1|\psi_2}$

Therefore $\mathrm{Tr}[(\mathcal{T}_A^{\psi_1|\psi_2})^n] = \mathrm{Tr}[X_A^n] = \sum_i \lambda_i^n$

$$S_A \equiv \lim_{n\to 1} S_A^{(n)} = -\sum_i \lambda_i \log \lambda_i$$

Introduction: Holographic pseudo-entropy

In AdS/CFT correspondence, pseudo entropy (PE) is dual to area of minimal surfaces in time-dependent Euclidean asymptotically AdS (aAdS) spaces



The picture is taken from arXiv: 2005.13801

aAdS side:

$$S_A = \min_{\substack{\Gamma_A \\ \partial\Gamma_A = \partial A}} \left[\frac{\mathrm{Area}[\Gamma_A]}{4G} \right]$$

$$\|$$

CFT side:

$$S_A = -\mathrm{Tr}[\mathcal{T}_A^{\psi_1|\psi_2} \log \mathcal{T}_A^{\psi_1|\psi_2}]$$

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Introduction: PRE in real time (Our focus)

What happens if we think about the pseudo-(Rényi) entropy in real-time?

i.e. we consider

$$\boxed{\mathcal{T}_A^{\psi_1|\psi_2}(t) = \frac{\mathrm{e}^{-iHt}|\psi_1\rangle\langle\psi_2|\mathrm{e}^{iHt}}{\langle\psi_2|\psi_1\rangle}}$$

$$\Downarrow$$

$$\boxed{S_A^{(n)}(t) = \frac{1}{1-n}\log \mathrm{Tr}[(\mathcal{T}_A^{\psi_1|\psi_2}(t))^n]}$$

Introduction: PRE in real time

One simple scheme in 2d CFTs is to consider two locally excited states:

Regulator

$$|\psi_i\rangle = e^{-\epsilon H}\mathcal{O}_i(t=0,x_i)|\Omega\rangle$$

Primay, descendent …

Vacuum

$$\mathcal{T}_A^{\psi_1|\psi_2}(t) = \frac{\mathrm{e}^{-iHt}|\psi_1\rangle\langle\psi_2|\mathrm{e}^{iHt}}{\langle\psi_2|\psi_1\rangle}$$

$$S_A^{(n)}(t) = \frac{1}{1-n}\log \mathrm{Tr}[(\mathcal{T}_A^{\psi_1|\psi_2}(t))^n]$$

Part 2:Pseudo-(Rényi) entropy for locally excited states in 2d CFTs

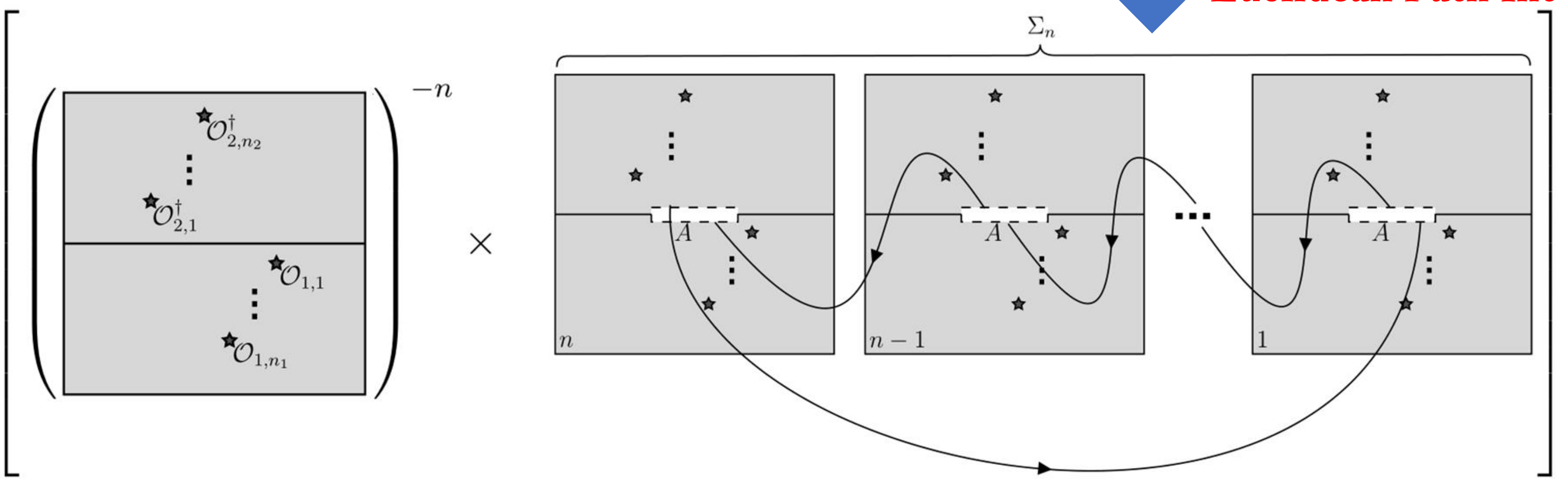
PRE for locally excited state: Replica trick

$$|\psi_j\rangle = \frac{1}{\sqrt{\mathcal{N}_j}}\mathcal{O}_{j,1}(-\tau_{j,1},x_{j,1})\mathcal{O}_{j,2}(-\tau_{j,2},x_{j,2})...\mathcal{O}_{j,n_j}(-\tau_{j,n_j},x_{j,n_j})|\Omega\rangle,$$
$$(\tau_{j,i+1}\geq\tau_{j,i}>0,\ i=1,2,...,n_j-1;\ j=1,2),$$

$$S_A^{(n)} = \frac{1}{1-n}\log \mathrm{Tr}[(\mathcal{T}_A^{\psi_1|\psi_2})^n]$$

Euclidean Path Integral

$$S_A^{(n)} = \frac{1}{1-n}\log$$



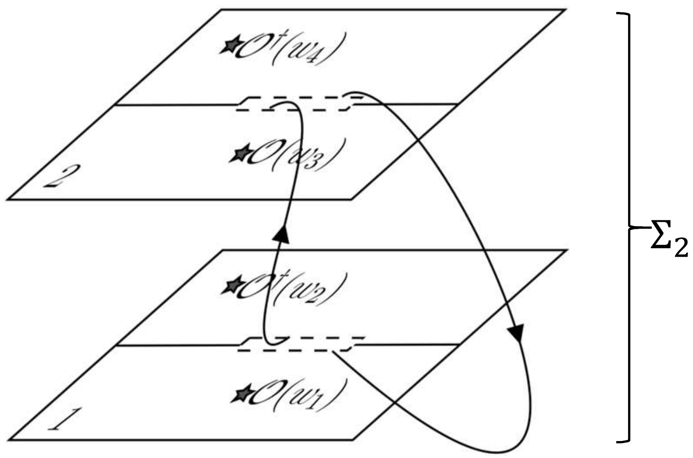
$$= S_{A;vac}^{(n)} + \frac{1}{1-n}\left(\underbrace{\log\left\langle(\mathcal{O}^{\dagger(n)}_{2,n_2}...\mathcal{O}^{(n)}_{1,n_1})...(\mathcal{O}^{\dagger(1)}_{2,n_2}...\mathcal{O}^{(1)}_{1,n_1})\right\rangle_{\Sigma_n}}_{(n_1+n_2)*n\text{-point function on }\Sigma_n} - n\log\underbrace{\left\langle\mathcal{O}^\dagger_{2,n_2}...\mathcal{O}_{1,n_1}\right\rangle_{\Sigma_1}}_{(n_1+n_2)\text{-point function on }\Sigma_1}\right).$$

PRE for locally excited state: Single primary

For $n = 2$, $n_1 = n_2 = 1$, $\Delta S_A^{(2)}$ is reduced to 4-point functions on Σ_2.

We further assume $\mathcal{O}_1 = \mathcal{O}_2 = \mathcal{O}$ to simplify the results

$$\Delta S_A^{(2)} = -\log \frac{\langle \mathcal{O}^{\dagger(2)}(\tau_2, x_2)\mathcal{O}^{(2)}(-\tau_1, x_1)\mathcal{O}^{\dagger(1)}(\tau_2, x_2)\mathcal{O}^{(1)}(-\tau_1, x_1)\rangle_{\Sigma_2}}{\langle \mathcal{O}^{\dagger}(\tau_2, x_2)\mathcal{O}(-\tau_1, x_1)\rangle_{\Sigma_1}^2}$$



Note1: Conformal map between Σ_2 and Σ_1

$$z = \left(\frac{w}{w-L}\right)^{1/n}, \quad (A = [0, L]),$$

$$z = w^{1/n}, \quad (A = [0, +\infty))$$

Note2: Analytic continuation of t

$$\tau_1 = \epsilon + it, \quad \tau_2 = \epsilon - it$$

PRE for locally excited state: Single primary

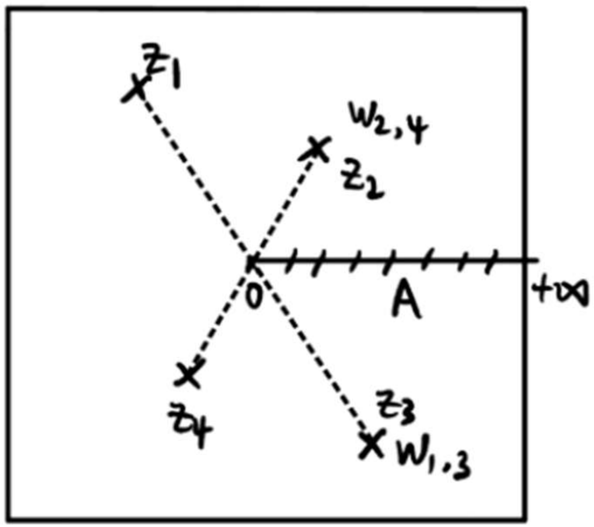
$$\Delta S_A^{(2)} = -\log \frac{\langle \mathcal{O}^{\dagger(2)}(\tau_2, x_2)\mathcal{O}^{(2)}(-\tau_1, x_1)\mathcal{O}^{\dagger(1)}(\tau_2, x_2)\mathcal{O}^{(1)}(-\tau_1, x_1)\rangle_{\Sigma_2}}{\langle \mathcal{O}^{\dagger}(\tau_2, x_2)\mathcal{O}(-\tau_1, x_1)\rangle_{\Sigma_1}^2}$$

$$z = \left(\frac{w}{w-L}\right)^{1/n}, \quad (A = [0, L]),$$
$$z = w^{1/n}, \quad (A = [0, +\infty))$$

$$(w_3, \bar{w}_3)_{\text{sheet 2}} = (w_1, \bar{w}_1)_{\text{sheet 1}} = (x_1 - i\tau_1, x_1 + i\tau_1)$$
$$(w_4, \bar{w}_4)_{\text{sheet 2}} = (w_2, \bar{w}_2)_{\text{sheet 1}} = (x_2 + i\tau_2, x_2 - i\tau_2)$$

$$\langle \phi_1(\vec{x}_1)\phi_2(\vec{x}_2)\phi_3(\vec{x}_3)\phi_4(\vec{x}_4)\rangle = f(\eta, \bar{\eta}) \prod_{i<j}^{4} z_{ij}^{\frac{h}{3} - h_i - h_j} \bar{z}_{ij}^{\frac{\bar{h}}{3} - \bar{h}_i - \bar{h}_j}$$

$$(\eta, \bar{\eta}) = \left(\frac{z_{12}z_{34}}{z_{13}z_{24}}, \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{13}\bar{z}_{24}}\right)$$



PRE for locally excited state: Single primary

$$\Delta S_A^{(2)} = -\log \frac{\langle \mathcal{O}^{\dagger(2)}(\tau_2, x_2)\mathcal{O}^{(2)}(-\tau_1, x_1)\mathcal{O}^{\dagger(1)}(\tau_2, x_2)\mathcal{O}^{(1)}(-\tau_1, x_1)\rangle_{\Sigma_2}}{\langle \mathcal{O}^{\dagger}(\tau_2, x_2)\mathcal{O}(-\tau_1, x_1)\rangle_{\Sigma_1}^2}$$

$$z = \left(\frac{w}{w-L}\right)^{\frac{1}{2}}, \quad A = [0, L]$$

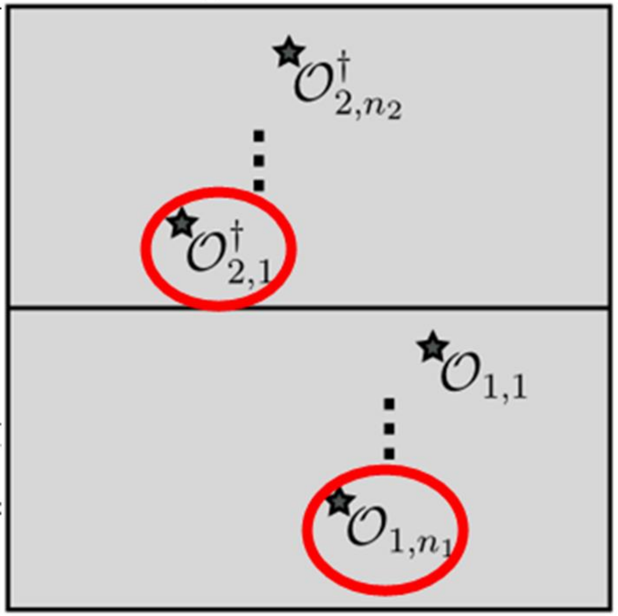
$$\boxed{z = w^{\frac{1}{2}},} \quad A = [0, +\infty)$$

$$\langle \mathcal{O}^{\dagger}(z_2, \bar{z}_2)\mathcal{O}(z_1, \bar{z}_1)\rangle_{\Sigma_1} = \frac{c_{12}}{|z_{12}|^{4\Delta_{\mathcal{O}}}}$$

$$\langle \mathcal{O}^{\dagger(2)}(\tau_2, x_2)\mathcal{O}^{(2)}(-\tau_1, x_1)\mathcal{O}^{\dagger(1)}(\tau_2, x_2)\mathcal{O}^{(1)}(-\tau_1, x_1)\rangle_{\Sigma_2} = |16 z_1^2 z_2^2|^{-4\Delta_{\mathcal{O}}} G(\eta, \bar{\eta})$$

$$\Delta S_A^{(2)} = \log \frac{c_{12}^2}{|\eta(1-\eta)|^{4\Delta_{\mathcal{O}}} \cdot G(\eta, \bar{\eta})}$$

PRE for locally excited state: Single primary



$$A = [0, \infty).$$

Table 1: Early time and late time behaviors of $(\eta, \bar{\eta})$ for the subsyster

$(\eta, \bar{\eta})$	$x_1 x_2 > 0$	$x_1 x_2 < 0$	
Late time $(t \to \infty)$	$(1, 0)$	$(1, 0)$	
Early time $(t \to 0)$	$(\frac{1}{2} + a, \frac{1}{2} + a)$	$x_1 > 0 > x_2$	$x_2 > 0 > x_1$
	$a = \frac{x_1 + x_2}{4\sqrt{x_1 x_2}}$	$(\frac{1}{2} + a, \frac{1}{2} - a)$	$(\frac{1}{2} - a, \frac{1}{2} + a)$

PRE for locally excited state: Single primary

Late time limit ($A = [0, \infty)$):

Quantum dimension of $\mathcal{O}$

Rational CFTs: $$\Delta S_A^{(2)} \simeq \begin{cases} 0, & t \to 0 \ \&\& \ x_1 \sim x_2, \\ \log d_{\mathcal{O}}, & t \to \infty. \end{cases}$$

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Large-cCFTs: $(c \to \infty)$ $$\mathrm{Re}\left[\Delta S_A^{(2)}\right] = 4\Delta_{\mathcal{O}} \log \frac{4t}{\sqrt{(x_1 - x_2)^2 + 4\epsilon^2}}$$

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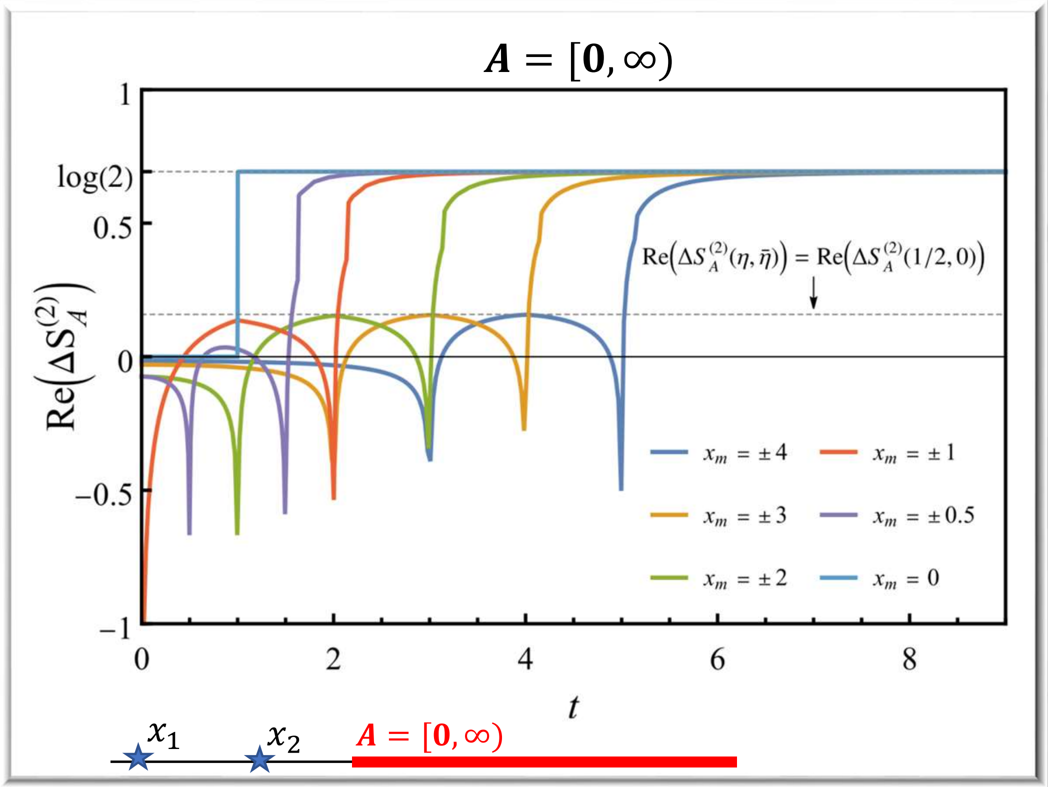
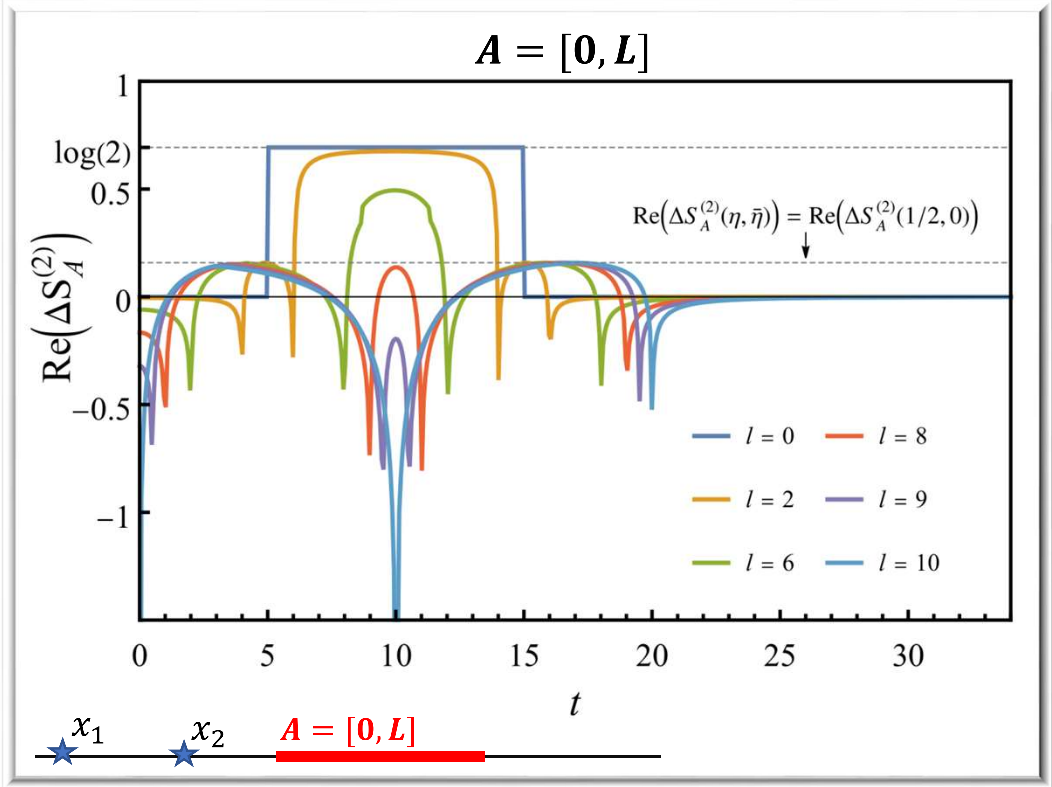
PRE for locally excited state: Single primary

Full-time evolution: $\boxed{\mathcal{O}=(e^{\frac{i}{2}\phi}+e^{-\frac{i}{2}\phi})}$-excitation in free scalar

$d_{\mathcal{O}}=2$

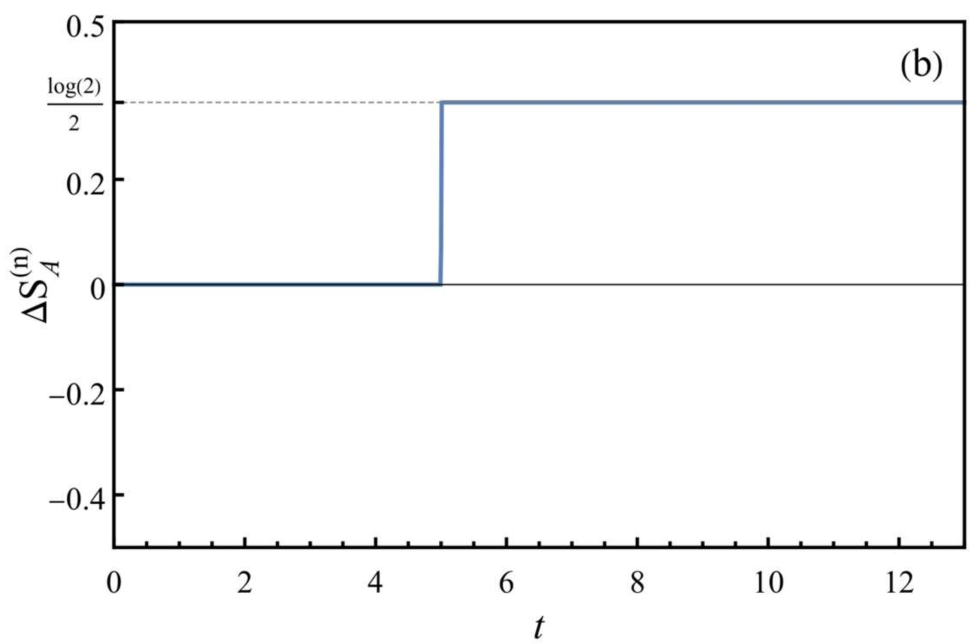




PRE for locally excited state: Single primary

When $A = [0, \infty)$, the late time limit of $\log d_{\mathcal{O}}$ is true for any order of $\Delta S_A^{(n)}$.

$$\lim_{t\to\infty} \Delta S_A^{(n)} = \log d_{\mathcal{O}}$$



Example:

σ(spin)-excitation in critical Ising model

$$\Delta S^{(n)}_{A=[0,\infty)}(x,-x,t) = \begin{cases} 0, & 0 \le t < |x|, \\ \log d, & t > |x|. \end{cases}$$

PRE for locally excited state: Linear combination

excited by linear combination operators

$$|\psi\rangle := \frac{1}{\sqrt{\langle \mathcal{O}^\dagger(x,\epsilon)\mathcal{O}(x,-\epsilon)\rangle}}\mathcal{O}(x,-\epsilon)|\Omega\rangle, \quad |\tilde{\psi}\rangle := \frac{1}{\sqrt{\langle \tilde{\mathcal{O}}^\dagger(\tilde{x},\epsilon)\tilde{\mathcal{O}}(\tilde{x},-\epsilon)\rangle}}\tilde{\mathcal{O}}(\tilde{x},-\epsilon)|\Omega\rangle,$$

$$\mathcal{O}(x,-\epsilon) = \sum_p C_p \mathcal{O}_p(x,-\epsilon), \quad \tilde{\mathcal{O}}(\tilde{x},-\epsilon) = \sum_p \tilde{C}_p \mathcal{O}_p(\tilde{x},-\epsilon).$$

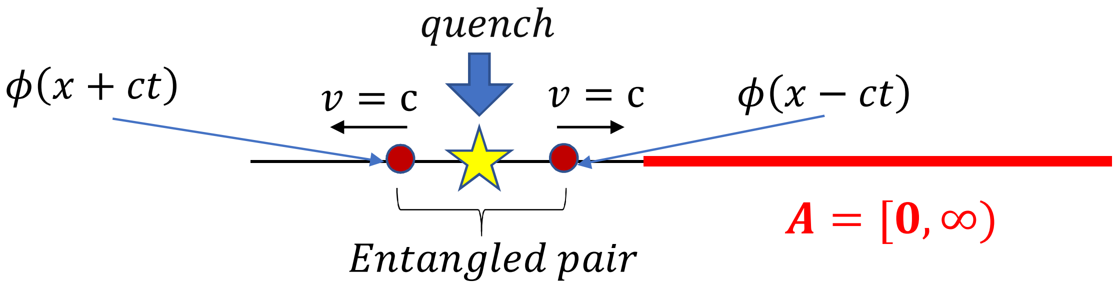
The expected late time limit ($A = [0,\infty)$)

2-point function of $\mathcal{O}$

$\log d_p$

$$\lim_{t\to\infty} \Delta S^{(n)}[\mathcal{T}_A^{\psi|\tilde{\psi}}(t)] = \frac{1}{1-n}\log\left[\sum_p \left(\frac{C_p\tilde{C}_p^*\langle \mathcal{O}_p^\dagger(\tilde{w},\bar{\tilde{w}})\mathcal{O}_p(w,\bar{w})\rangle}{\sum_{p'} C_{p'}\tilde{C}_{p'}^*\langle \mathcal{O}_{p'}^\dagger(\tilde{w},\bar{\tilde{w}})\mathcal{O}_{p'}(w,\bar{w})\rangle}\right)^n e^{(1-n)S^{(n)}[\mathcal{O}_p]}\right]$$

The expected late time limit of $\Delta S_A^{(n)}$



The EE of A contains only the contribution of the right-moving mode as t goes to infinity

$$|\mathcal{O}_p(x)\rangle = \sum_i a_i^p(x)|p_i(x)\rangle \otimes |(\bar{p}_i(x)\rangle \qquad (H = \bigoplus_p H_p \bigotimes H_{\bar{p}})$$

Schmidt decomposition

Sub-Verma module

$$S^{(n)}[\mathcal{O}_p(x)] = \frac{1}{1-n}\log\left\{\mathrm{Tr}_{(\oplus_p H_p)}\left[\left(\mathrm{Tr}_{(\oplus_p H_{\bar{p}})}|\mathcal{O}_p(x)\rangle\langle\mathcal{O}_p(x)|\right)^n\right]\right\} = \boxed{\frac{1}{1-n}\log\sum_i\left(a_i^p(x)\right)^{2n}}$$

PRE for locally excited state: Linear combination

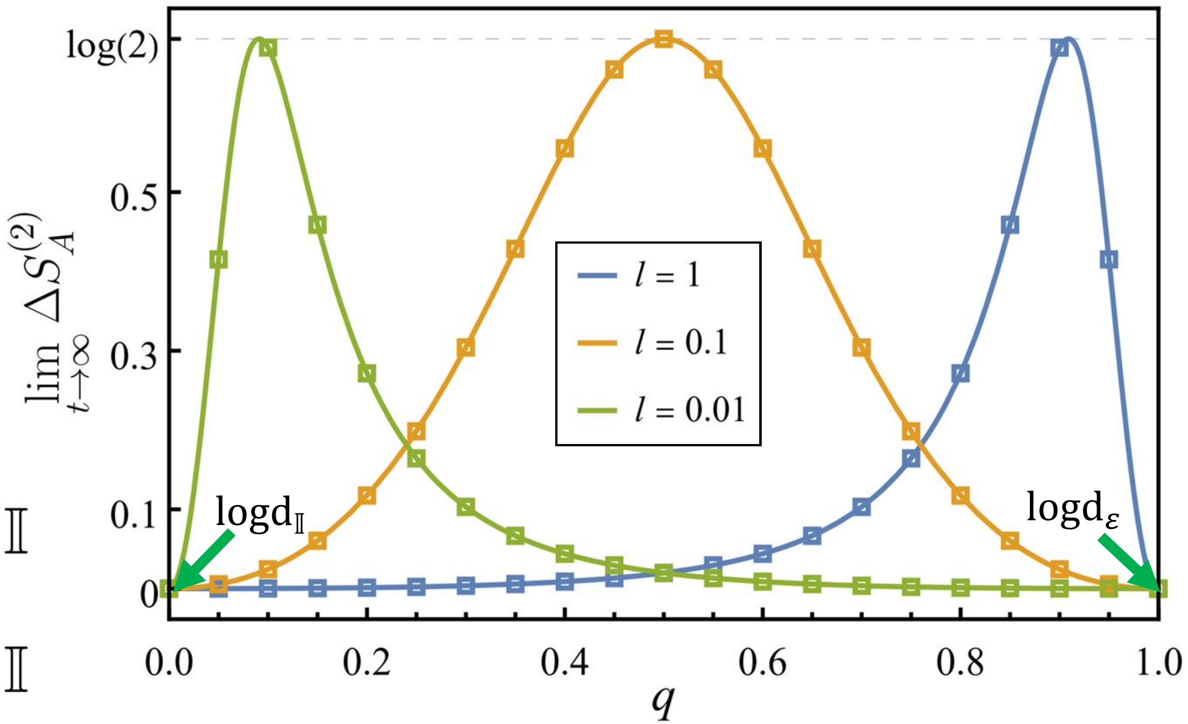
Verify it with numerical results: $\varepsilon + \mathbb{I}$ in critical Ising model

$$\varepsilon \times \varepsilon = \mathbb{I}, \quad \boxed{\sigma \times \sigma = \mathbb{I} + \varepsilon,} \quad \sigma \times \varepsilon = \sigma.$$

$$q = \tilde{q}, \quad |\tilde{w} - w| = l$$

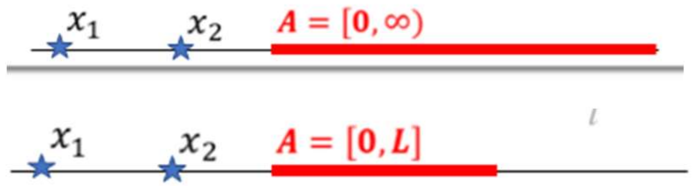
$$\tilde{\mathcal{O}}(\tilde{w}, \bar{\tilde{w}}) = \tilde{q} \cdot \varepsilon(\tilde{w}, \bar{\tilde{w}}) + (1 - \tilde{q}) \cdot \mathbb{I}$$

$$\mathcal{O}(w, \bar{w}) = q \cdot \varepsilon(w, \bar{w}) + (1 - q) \cdot \mathbb{I}$$



Hidden symmetry of PRE

$$\eta(x_2,x_1,t)=\left[\eta(x_1,x_2,t)\right]^*,\quad \bar{\eta}(x_2,x_1,t)=\left[\bar{\eta}(x_1,x_2,t)\right]^*,$$
$$\eta(-x_1,-x_2,t)=1-\bar{\eta}(x_1,x_2,t),\qquad (A=[0,\infty)),$$
$$\eta(L-x_1,L-x_2,t)=\bar{\eta}(x_1,x_2,t),\qquad (A=[0,L]),$$



For diagonal CFTs:

$$G(\eta,\bar{\eta})=G(\bar{\eta},\eta),$$
$$G(\eta^*,\bar{\eta}^*)=[G(\eta,\bar{\eta})]^*.$$

Based on On-going work, Psuedo Hermitian vs PRE

$$\Delta S^{(2)}_{[0,L]}(x_1,x_2,t)=\Delta S^{(2)}_{[0,L]}(L-x_1,L-x_2,t),$$
$$\Delta S^{(2)}_{[0,\infty)}(x_1,x_2,t)=\Delta S^{(2)}_{[0,\infty)}(-x_1,-x_2,t).$$

Part 3: Summary

Summary

- We obtain the full-time evolution picture of pseudo-(Rényi) entropy for locally excited states.

- We obtain several limiting behaviors of $\Delta S_A^{(n)}$. ($\log d_{\mathcal{O}}$bound for rational CFTs)

- We find a interesting insertion configuration of operators, for example: $x_1 = -x_2$ for $A = [0, \infty)$, in which the nth pseudo-(Rényi) entropy may behaves like Rényi entropy.

- We obtain the late time limits of the nth pseudo-(Rényi) entropy for linear combination operators, which are in good agreement with numerical examinations.

Thanks for your attention