Completeness of spin chain via Bethe/Gauge correspondence

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Based on the work with Peng Zhao, Rui-Dong Zhu and Hao Zou
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- Thanks to duality/correspondence/non-perturbative method, many exact results are known in string theory and QFT.
- e.g. Seiberg-Witten theory of $\mathcal{N}=2$ SYM, localization, ...
- What can we learn from these results?
- These results are more useful than we thought.
- It is time to use string theory/QFT to answer some old questions.

Heisenberg $XXX_{1/2}$ Spin chain

$$H = \sum_{n=1}^{L} \vec{S}_n \cdot \vec{S}_{n+1} - \frac{1}{4}, \quad \vec{S}_{L+1} = \vec{S}_1$$

The periodic case can be solved exactly by using the Bethe ansatz equations (BAEs) [Bethe' 31]. See also Yunfeng's and Yi's talk.

$$\left(\frac{\lambda_j + i/2}{\lambda_j - i/2}\right)^L = -\prod_{k=1}^{M} \frac{\lambda_j - \lambda_k + i}{\lambda_j - \lambda_k - i}, \quad j = 1, \cdots, M.$$

- Solving the BAEs, we can determine the eigenvalue and eigenstate:

$$E_M = -\sum_{j=1}^{M} \frac{1/2}{\lambda_j^2 + 1/4}$$

$$|\lambda_1, \lambda_2, \cdots, \lambda_M\rangle \sim B(\lambda_1) B(\lambda_2) \cdots B(\lambda_M) |\uparrow\uparrow \cdots \uparrow\rangle$$

How many physical states?

- $\#$ of solutions of BAEs.
- Warn: not all solutions will provide physical state.
- Singular solutions, $\{\lambda_1 = \frac{i}{2}, \lambda_2 = -\frac{i}{2}, \lambda_3, \cdots\}$, provide a trivial solution to the BAEs. The energy will be divergent.
- After regularization, some of the singular solutions provide physical states, say singular physical solution [Nepomechie-Wang '13].
- The number of the physical states are conjectured by [Hao-Nepomechie-Sommese '13]

$$\mathcal{N}_{\text{reg}} + \mathcal{N}_{sp} = \binom{L}{M} - \binom{L}{M-1}$$

- This conjecture has also been tested numerically by using Gröbner basis in algebraic geometry [Jiang-Zhang '17].

- In the case of higher spin, one has to include strange solution with repeated roots [Hao-Nepomechie-Sommese '13]

$$\mathcal{N}_{\mathrm{reg}} + \mathcal{N}_{sp} + \mathcal{N}_{str} = c_s(L, M) - c_s(L, M-1)$$

- The higher rank spin chain is solved by using the nested BAEs, whose solution counting is still not well studied.

String theory/QFT: "Witten index"

Bethe/Gauge correspondence [Nekrasov-Shatashivili'09]

Twisted boundary condition: $S^{\pm}_{L+1} = S^{\pm}_{1} \to S^{\pm}_{L+1} = e^{\mp i\theta} S^{\pm}_{1}$.

$$XXX_{s,\theta}: \quad e^{i\theta}\left(\frac{\lambda_j + is}{\lambda_j - is}\right)^L = -\prod_{k=1}^{M} \frac{\lambda_j - \lambda_k + i}{\lambda_j - \lambda_k - i}$$

Low energy of 2D $\mathcal{N} = (2,2)$ gauge with $L_f, L_{\bar{f}}$ and adjoint matters.

- BAE of XXX spin chain $\leftrightarrow$ vacuum equation on Coulomb branch
- Length $L \leftrightarrow L_f = L_{\bar{f}} = L$ flavour
- Magnon M $\leftrightarrow$ Gauge group $U(M)$
- Yang-Yang function $\Longleftrightarrow$ effective twisted superpotential
- twisted parameter $\theta \Longleftrightarrow$ complex coupling $\tau = ir + \theta/2\pi$
- spin and inhomogeneity of each site $\Longleftrightarrow$ mass m_f and $m_{\bar{f}}$

Witten index

Elliptic genera[Benini et al '13]:

$$Z_{T^2} \sim \sum_{u^*} \oint_{\mathrm{JK},u^*} d^M u \, Z_{1-\mathrm{loop}}, \;\; Z_{1-\mathrm{loop}} = Z_{\mathrm{vec}} Z_{\mathrm{adj}} Z_f Z_{\bar{f}}$$

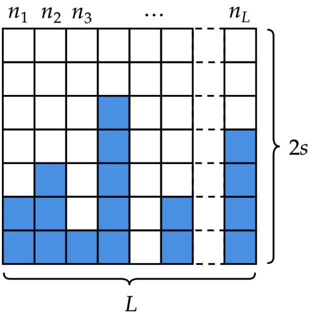
$$Z_{\mathrm{adj}} = \prod_{i,j=1}^{M} \frac{\theta_1(\tau|u_{ij}+\lambda-z)}{\theta_1(\tau|u_{ij}+\lambda)}, \quad Z_{\mathrm{fund}} = \prod_{i=1}^{M} \left(\prod_{\alpha=1}^{L} \frac{\theta_1(\tau|u_i-\xi_\alpha+\chi-z)}{\theta_1(\tau|u_i-\xi_\alpha+\chi)} \right)$$

- Only poles at $u_{ij}+\lambda=0$ and $u_{ij}-\xi_\alpha+\chi=0$ contribute.
- The choice of the pole: $\{n_1, n_2, \cdots, n_L\}$ with $\sum n_\alpha = M$

$$u^*_{i_{m_\alpha}+1} = \xi_\alpha - \chi - m_\alpha \lambda, \quad m_\alpha = 0, \ldots, n_\alpha - 1$$

- Witten index: $\lim_{z\to 0} Z_{T^2} =$

$$\lim_{z\to 0} \sum_{\vec{n}} \prod_{\alpha=1}^{L} \prod_{m_\alpha=1}^{n_\alpha-1} \frac{\theta_1\left(\tau|(m_\alpha-2s)\lambda\right)}{\theta_1\left(\tau|(m_\alpha-2s)\lambda+z\right)} \cdots = \sum_{\substack{\vec{n} \\ \text{s.t. } n_\alpha \le \min\{M,2s\}}} 1$$



- $2s = 1$: $\mathcal{N}_{\text{phys}} = \binom{L}{M}$
- $2s \geq 1$:

$$\mathcal{N}_{\text{phys}} = c_s(L, M) = \sum_{j=0}^{L} (-1)^j \binom{L}{j} \binom{L + M - 1 - (2s+1)j}{L - 1}$$

The index of higher rank spin chian/Quiver gauge theory can be computed in a similar way.

Witten index $\to \mathcal{N}_{phys}$ of $XXX_{s,\theta}$

Tested numerically with generic θ by

- Solving the BAEs directly
- Q-system (Thanks to Yunfeng)
- Gröbner basis

For some special θ, we find singular physical solution.

$(L, M, s) = (4, 3, 1)$ and $\theta = \frac{2\pi}{3}$

$\mathcal{N}_{\text{phys}} = 16$, where $\mathcal{N}_{\text{reg}} = 15$ and $\mathcal{N}_{sp} = 1$.

When $\theta \to 0$, some of regular solutions will go to infinity.

Untwisting limit of $XXX_{1/2}$

- Taking the limit $\theta \to 0$, a $SU(2)$ symmetry is enhanced

$$[\mathbf{S}_z, H] = 0 \xrightarrow{\theta \to 0} [\mathbf{S}_{z,\pm}, H] = 0, \quad \vec{\mathbf{S}} = \sum_{i=1}^{L} \vec{S}_i$$

- At this limit, only the highest weights contribute to the eigenstate: $\mathbf{S}_+|\Psi\rangle = 0$ [Faddeev-Takhtadzhyan '81].

- We should discard the state $\mathbf{S}_+|\Psi\rangle \neq 0$, e.g. $S_-^n|\Psi\rangle_{M-n}^{HW}$.

$$\lim_{\theta \to 0} \mathcal{N}_{\rm phys} = \binom{L}{M} - \binom{L}{M-1}$$

Discussion and Future work

- Witten index counts the physical states of $XXX_{s,\theta}$ spin chain.
- Duality of $\mathcal{N} = (2, 2)$ gauge: same # states and same elliptic genera, e.g. $U(M)$+L flavours $\leftrightarrow$ $U(2sL - M)$+L flavours.

$$\Updownarrow$$

- The Weyl transform of the QQ-relation leads to the dual BAEs.
- Also works for higher rank spin chain(nested BAEs)/Quiver gauge

- Formula for the number of singular physical solution/strange solution?
- More invariants from the expansion of index?
- Open spin chain $\leftrightarrow$ 2D gauge with O-type gauge group
- XXZ spin chain $\leftrightarrow$ 3D gauge theory

More lessons from String theory/QFT?

Thanks for your attention !